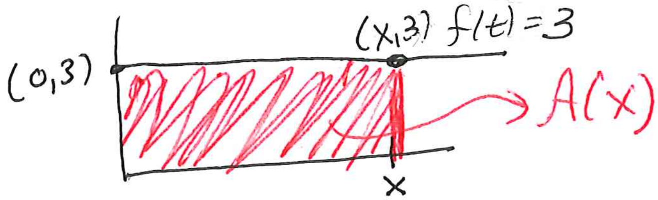
Area Accumulation Functions and the Fundamental Theorem of Calculus
Homework

1. Suppose $f(t)=3$. Let $A(x)=\int_0^x f(t)\,dt$ for $x \geq 0$.

   a. Mark an $x>0$ on a graph of f. Shade the area that corresponds to $A(x)$.



   b. Use geometry to help you compute the value of A(x) shown in the diagram in part a.

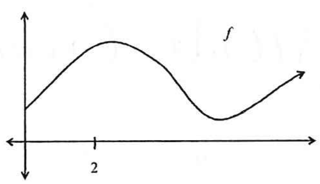
This is a rectangle with height 3 and width x

$A(x) = 3x$

   c. Show that f is, indeed, the derivative of the function you got in part b.

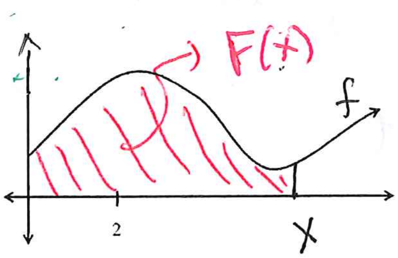
$A'(x) = 3 = f(x)$

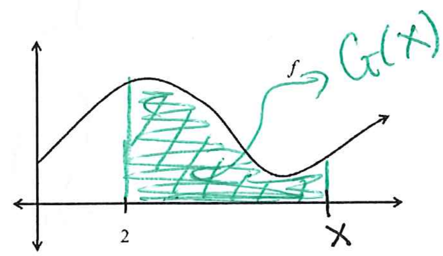
2. Consider the graph of the function f, shown below. Define two area accumulation functions for f:



$$F(x)=\int_0^x f(t)\,dt \text{ on } [0,\infty) \text{ and } G(x)=\int_2^x f(t)\,dt \text{ on } [2,\infty)$$

   a. On the graphs below, add details that give geometric interpretations for $F(x)$ and $G(x)$, respectively Be sure to label your diagrams so that it is clear what you intend.





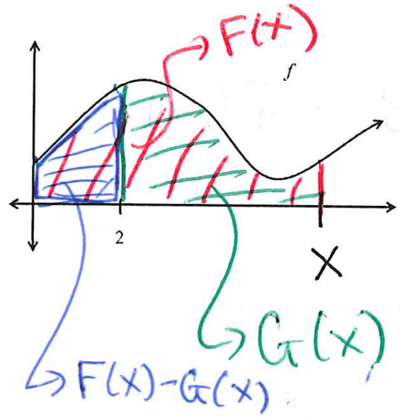
b. Determine a value of x for which $F(x) = 0$. Use properties of integrals to justify your choice. (Hint: Which property shown in Table 5.3 of B&C is relevant here?)

$$F(0) = \int_0^0 f(t)\,dt = 0$$

c. Determine a value of x for which $G(x) = 0$. Briefly, give a reason for your choice.

$$G(2) = \int_2^2 f(t)\,dt = 0$$

d. On the graph below, show the constant by which F and G differ at some $x > 2$.



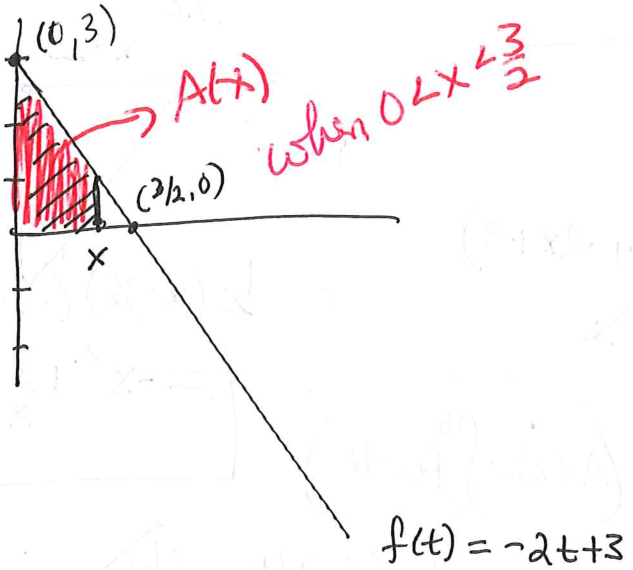
e. Use properties of integrals to show analytically that F and G differ by this constant.

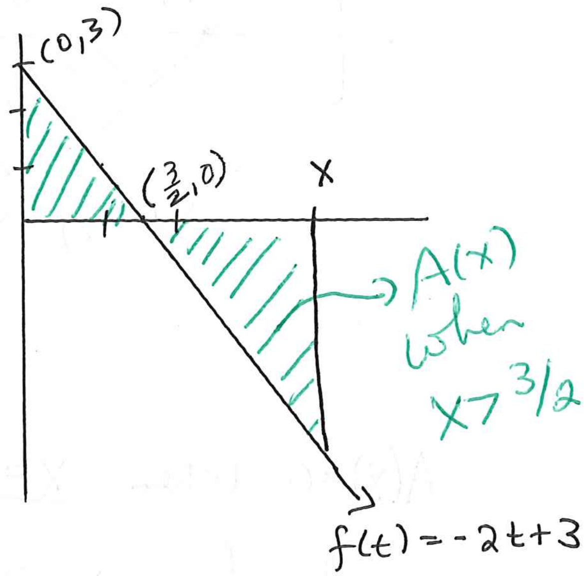
$$F(x) - G(x) = \int_0^x f(t)\,dt - \int_2^x f(t)\,dt$$

$$= \int_0^2 f(t)\,dt$$

3. Suppose $f(t) = -2t + 3$. Let $A(x) = \int_0^x f(t)\,dt$ for $x \geq 0$ and $F(x) = \int_{3/2}^x f(t)\,dt$ for $x \geq 3/2$ be two different area accumulation functions for f.

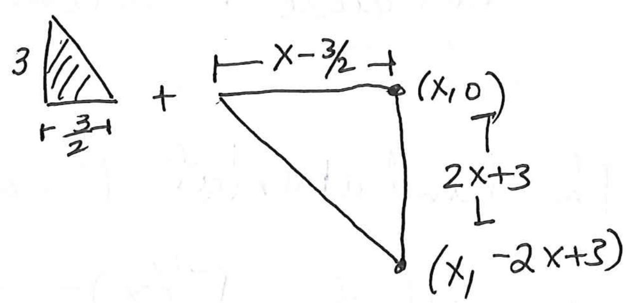
   a. Draw two diagrams that illustrate how we get values $A(x) = \int_0^x f(t)\,dt$: one should show $A(x)$ when $0 \leq x < \frac{3}{2}$ and the other should show $A(x)$ when $x > \frac{3}{2}$.





   b. Use geometry to help you find a formula for $A(x) = \int_0^x f(t)\,dt$ when $x > \frac{3}{2}$. Draw a picture to help you with (and to explain!) your calculation. Label the picture carefully with information relevant to the calculation.

$$A(x) = \int_0^x f(t)\,dt =$$

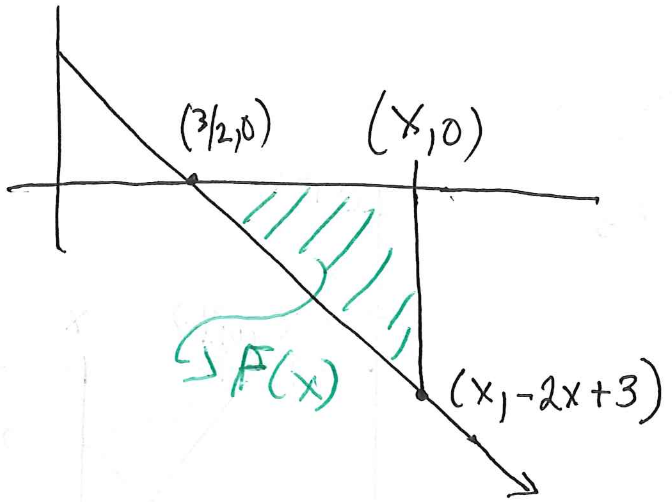


$$= \tfrac{1}{2}\left(\tfrac{3}{2}\right)(3) + \tfrac{1}{2}\left(x - \tfrac{3}{2}\right)(-2x+3)$$

$$= \frac{9}{4} + \frac{1}{2}\left(-2x^2 + 6x - \frac{9}{2}\right)$$

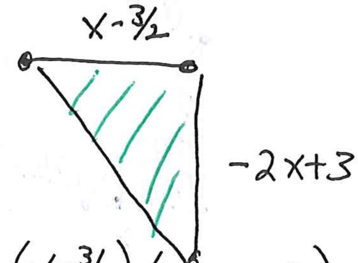
$$= \frac{9}{4} + x^2 + 3x - \frac{9}{4} = -x^2 + 3x$$

c. Use geometry to find a formula for $F(x)=\int_{3/2}^{x} f(t)\,dt$. Draw a carefully labeled diagram that helps you (and illustrates!) your calculation.



$$F(x)=\int_{3/2}^{x} f(t)\,dt$$

=

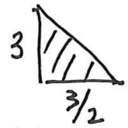


$$=\tfrac{1}{2}\left(x-\tfrac{3}{2}\right)(-2x+3)$$

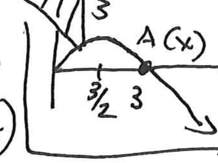
$$=-x^2+3x-\frac{9}{4}$$

d. For what values of x is $A(x) = 0$? For what values of x is $F(x) = 0$?

$A(x)=0$ when $\underline{x=0}$, $\left(A(x)=\int_0^0 f(t)\,dt\right)$

and when $\underline{x=3}$, $A(3)=\int_0^3 f(t)\,dt =$  $+$  $=0$.

(A increases from 0 to 3/2 and decreases thereafter)



$F(x)=0$ only when $x=3/2$ $\left(F(3/2)=\int_{3/2}^{3/2} f(t)\,dt\right)$

because F decreases throughout its domain.

e. Use the Fundamental Theorem of Calculus to compute a formula for $F(x)$. (Hint: be sure that the formula you get is 0 at the value you specified in part d.)

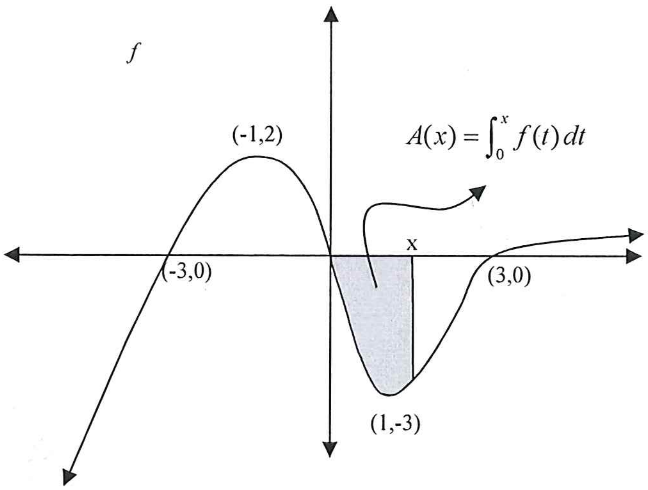
The fundamental theorem of Calculus tells us that $F'(x)=f(x)=-2x+3$. We also know that $F(3/2)=0$.

So $F(x)=-x^2+3x+C$ & $0=F(3/2)=-(3/2)^2+3\left(\frac{3}{2}\right)+C$

So $C=\frac{9}{4}-\frac{9}{2}=-\frac{9}{4}$.

$$\boxed{F(x)=-x^3+3x-\frac{9}{4}}$$

4. Consider the function f and its area accumulation function $A(x) = \int_0^x f(t)\,dt$, shown below:



Answer the following questions about A. Give a brief explanation for each of your answers.

a. According to the Fundamental Theorem of Calculus, what is the relationship between the function A and the function f? The FTC tells us that

$$A'(x) = f(x).$$

b. On what interval(s) is A increasing? On what interval(s) is A decreasing?

A is increasing where $A'(x) = f(x)$ is positive. This is on $[-3, 0]$ and on $[3, \infty)$. Likewise, A is decreasing when f is negative: on $(-\infty, -3]$ and on $[0, 3]$

c. Does A have any local maxima or minima? If so, where?

Part b gives us the information summarized in the chart:

A'	$-$		$+$		$-$		$+$
A	$\searrow$	-3	$\nearrow$	0	$\searrow$	3	$\nearrow$

$x=-3$ is a local min
$x=0$ is a local max
$x=3$ is a local min.

So A has local mins at $x = \pm 3$ and a local max at $x = 0$

d. On what interval(s) is A concave up? On what interval(s) is A concave down?

A is concave up where $A' = f$ is increasing: on $(-\infty, -1)$ and on $(1, \infty)$. A is concave down on $(-1, 1)$ because A' is decreasing there.