Partial Differential Equations

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The junior-senior level Partial Differential Equations (PDEs) course serves both as a first introduction to serious applied mathematics and as the "most natural" continuation of the four-semester calculus sequence (including introductory ODEs). It may be taught as a full year course or as a one-semester course. In this report, we focus on a one-semester course and only briefly consider possible follow-up courses.

Course Goals

While PDEs are used widely by engineers, scientists, physicists and applied mathematicians to model physical phenomena, it is rare to encounter a simple enough problem that can be attacked by the analytic techniques learned in a first-semester PDEs course. It's just a sad fact that "real-world" PDE problems are often nonlinear and/or involve complicated geometry that put them out of reach of these basic analytic techniques. Nevertheless, students need to learn about basic linear PDEs because more complicated PDEs often have behaviors similar to those of basic PDEs, and they need to learn these basic analytic techniques because many other techniques rely or build on these. So, the overall goal for a first-semester course on applied PDEs should be to give students an appreciation for the big world of PDEs and their applications by encouraging them to model using PDEs, while teaching students about a canonical set of PDEs and solution techniques.

Students in this course should have facility with calculus and solving ODEs. A PDE course provides students with the opportunity to see mathematics used to solve real problems, giving them a better appreciation of mathematics and a greater context in which to understand its importance. After completing the course, students' deeper appreciation of calculus can in turn enhance their understanding of science and the role that mathematics plays in its development, helping them to learn to communicate with scientists and/or engineers, in industry or at a research center.

Consequently, this course is taught rather differently than other courses at this level, such as algebra or real analysis or topology. The goal is not to proceed slowly and make sure the students can write out complete rigorous proofs. The goal is to cover a larger amount of material, in the context of a bigger scientific picture. This implies less emphasis on proof, but care, honesty and depth should not be sacrificed. We make the distinction between rigor and proof: one can still be rigorous, but not prove all of the results that are required. For example, it is important to mention that interchanging infinite sum and integration often requires uniform convergence, but one could skip the proofs. Also, if students have some background with linear algebra (especially inner product spaces), it is beneficial to make the connection between differential operators and matrices. Careful heuristic arguments are commonplace in these courses, more detailed than the heuristic arguments normally used in calculus.

This connection to science and careful approach to building heuristic arguments should prepare the successful student to take other applied mathematics courses (in the Mathematics Department or in Economics, Engineering, Chemistry, etc.) and motivate her to take, e.g., graduate real analysis, having seen the need to understand the L^p spaces (for p=1,2,∞) and their completeness. For instance, the L^1 and L^∞ norms of the nonnegative solution $u(t,.)$ of the heat equation give the total heat content and the maximum temperature at time $t > 0$. The square of the L^2 norm of various functions of the solution of the wave equation give the kinetic, potential and total energies at time *t*. The solutions are constructed using infinite series of functions, integrals, or successive approximations, whence completeness of the underlying normed vector space is needed to obtain existence results.

Concepts and Topics

What are the key concepts of PDE that a student should see in a one-semester course? (If one knows a student will take a second semester, then some of these topics can be deferred until then, of course.) We start by describing the key concepts in a one-semester course, which should inform and relate to all of the topics. We then suggest topics for such a course and consider additional topics that could be covered in a second semester.

Concept #1. How PDEs are derived. If the equations are important, why is that so and where do they come from? Traditionally they come from Physics, but lately Biology and Economics/Finance have become big PDEs customers. The mathematics texts are often weak here. Some give "explanations" that are so brief as to be useless and unbelievable. Some are too detailed and take too long. The teacher has an obligation to be competent enough in the applied areas to help students navigate the deficiencies in the text. Feynman's "undergraduate" Lectures remain a clear and inspirational source of information for mathematics students and teachers. The wave equation deals with energy and L^2 spaces, but for the heat equation, the natural "norms" are the total heat content and the maximum temperature, which correspond to the L^1 and L^∞ norms. An introductory course on PDEs should introduce students to some fundamental PDEs: first order linear PDEs (and how they relate to conservation laws) and the three canonical second order PDEs (heat, wave, Laplace/Poisson).

Concept #2. Fourier Series. For the heat and wave equations in one space dimension, on bounded intervals, separation of variables is a most valuable tool, because it reduces everything to the solution of second order constant coefficient ODEs, which students should already know how to solve. This leads to inner products, complex numbers (for genuine simplification), the "right way" to redo plane geometry, orthonormal bases, and completion (and thus Hilbert space). By viewing the Laplacian $L=d^2/dx^2$ as an operator, selfadjointness, eigenvalues and eigenvectors (eigenfunctions) arise.

Concept #3. "Explicit" formulas for the solutions of a few problems. Examples of such formulas include d'Alembert's formula for the initial value problem for the wave equation on the line, Gauss' integral formula for the initial value problem for the heat equation on the line, and Poisson's integral formula for the solution of the Dirichlet problem for harmonic functions in the unit disk in the plane. The formulas show that the wave [resp., heat] problem has a finite [resp., infinite] speed of propagation, and the parabolic and elliptic problems have solutions with C^∞ (and even analytic) smoothness away from the "boundaries", while the hyperbolic wave equation preserves regularity rather than increasing it. These are big issues for the more complicated versions of these equations, and this provides some motivation for studying advanced tools later (e.g. Sobolev spaces).

Concepts #4, #5. Maximum Principle and energy methods. The Maximum Principle studies properties of second order elliptic and parabolic equations by means of the first and second derivative tests. Energy methods mean (in the simplest case) take an equation, multiply it by some function, and then integrate it. Integration by parts (or the Divergence Theorem) often leads to wonderful conclusions, for both science and mathematics. At least a little bit of these topics can be studied in the first semester PDEs course.

The following concepts can be woven through multiple topics covered in the course. Such topics typically include:

- First-order PDEs: their derivation (usually conservation laws) and basic behavior and uses.
- Canonical second-order linear PDEs (heat, wave, Laplace): their derivation and basic behavior and uses.
- Boundary and initial conditions, well-posed problems
- Solution techniques:
    - method of characteristics, traveling waves
    - separation of variables and Sturm-Liouville eigenvalue problems, eigenfunction series
    - integral transforms (approach of inversion depends on whether students have experience with complex analysis)
    - similarity solutions
    - Green's functions and the method of images

Notice that most of these solution techniques rely on turning PDEs into ODEs, so this course builds on students' understanding of and facility with ODEs. If students have experience with Frobenius series, one could also do separation of variables in different geometries and make use of Bessel functions, Legendre polynomials, etc.

Finally, a first course on PDEs should expose students to a wide range of applications of PDEs, and preferably to specific models that can be attacked using the techniques above. Here is a partial list of some application fields that are suited for a first course in PDEs:

Applications to:

- Engineering: heat/diffusion equation for heat/mass transfer, wave equation for motion of strings, membranes.
- Physics: Laplace/Poisson equation for electrostatic potential, Schroedinger equation (quantum bound states of the hydrogen atom).
- Biology: reaction diffusion equations, models of swarming (integro-differential equations).
- Economics: Black-Scholes PDE for stock options, fair strategies for bonds.

There are many other applications of PDEs that one can work into a course. The examples above are some of those that can be reached through the solution techniques described above. When discussing applications, it is natural to tell students about some nonlinear PDEs and other PDEs that come from different disciplines. This area of applied mathematics is rapidly evolving, and new examples of PDEs should be shown to students, such as the in-painting using PDEs, or modeling of swarms using nonlocal PDEs.

This may be an appropriate amount of material for the first semester, and maybe not, depending on the teacher, but a second semester provides an opportunity to complete any topics that were missed, to add more topics, and to build on those already covered. One may consider extensions of Fourier series, such as Fourier transforms (when a bounded interval or circle is replaced by the whole line) and the Mellin transform (the analogue to the Fourier transform obtained by replacing the group $(\mathbb{R},+)$ by its exponential image, the positive real numbers under multiplication). The wave equation in three dimensions (and Huygens' Principle) and numerical studies (finite differences and finite elements), and nonlinear first order PDEs (conservation laws, the Hamilton-Jacobi equation) are also appropriate topics for a second semester.

Technology

There are several ways that technology can improve the teaching and learning of PDEs. First, computer-algebra systems have evolved to the point where they are easy to use and can greatly aid in the solution of PDEs. While they're still not great for directly solving PDEs, they can help simplify nasty algebra, compute tedious integrals, compute solutions of ODEs, and help us visualize solutions either through graphs or animations. Computer-algebra systems can be incorporated into courses in various ways. Some instructors may choose to use them only for class demos and require students to do things by hand, or they might include training on these systems and require students to use them for their own work.

Second, there are a variety of ways that technology can help students visualize and gain an intuition for PDEs and the physical systems that they describe. For example, there are applets to visualize the electron orbitals of the hydrogen atom, movies of Chladni plates, movies of water surface waves, etc. When the analytic solution to a PDE is not available, numerical techniques can be helpful to show the behavior of the PDE.

Student Outcomes

Based on these recommendations, at the end of a one-semester PDE course, we expect that students:

- See PDEs as useful tools for describing and modeling a vast range of physical phenomena.
- Understand that linear PDEs are often solved by identifying the correct differential operator and solving the associated eigenvalue problem.
- Can use their understanding of the behavior of basic, canonical PDEs to build an intuition for how more complicated PDEs will behave, and to guess at the underlying PDE for a given physical phenomenon.
- Know the appropriate solution strategy for a particular PDE problem and can correctly carry it out.

A PDEs course has the potential to act as a bridge from calculus and introductory ODEs to the bigger world of applied mathematics. Students can gain a deeper appreciation of the connections between mathematics and sciences/economics, as well as the motivation to continue their studies in mathematics.

Textbooks

Remark*: The presence of a text on this list is not meant to imply an endorsement of that text, nor is the absence of a particular text from the list meant to be an anti-endorsement. The texts are chosen to illustrate the sorts of texts that support an undergraduate PDEs course.*

1. Asmar, N. H., *Partial Differential Equations with Fourier Series and Boundary Value Problems*, Pearson Prentice Hall, 2005.

2. Broman, A., *Introduction to Partial Differential Equations: From Fourier Series to Boundary Value Problems*, Courier Corporation, 2012.

3. Cooper, J. M., *Introduction to Partial Differential Equations with MATLAB*, Springer Science and Business Media, 2012.

4. Costanda, C., *Solution Techniques for Elementary Partial Differential Equations*, CRC Press, 2002.

5. Farlow, S., *Introduction to Partial Differential Equations for Scientists and Engineers*, Courier Corporation, 2012.

6. Haberman, R., *Applied Introduction to Partial Differential Equations with Fourier Series and Boundary Value Problems*, Pearson, 2013.

7. Olver, P., *Introduction to Partial Differential Equations*, Springer Science and Business Media, 2013.

8. Pinsky, M. A., *Partial Differential Equations and Boundary Value Problems with Applications*, 3rd Ed, American Mathematical Society, 2011.

9. Strauss, W. A., *Partial Differential Equations: An Introduction*, Wiley, 2007.

10. Myint-U, Tyn, and L. Debnath, *Linear Partial Differential Equations for Scientists and Engineers*, Springer Science and Business Media, 2007.