A Proof that π is Irrational

The following proof requires no calculus beyond what someone would see in Calc A here at Kenyon or a first high school calculus course. The proof is due in essence to Ivan Niven; I've based this presentation on an outline by Helmut Richter.

Theorem: π is irrational.

Proof: Assume for the sake of a contradiction that $\pi = \dfrac{a}{b}$ for whole numbers a and b.

We are going to proceed – for reasons that will be clear by the end, but not much before – by defining a function dependent on a number n. What that number n is doesn't matter for now. The initial portions of the argument work for any whole number n, and we'll figure out what particular n we wanted later. (There will eventually be a fraction with an n in the denominator that we want to make small, so we'll be choosing any n big enough to accomplish this task.)

For now, whatever n is, define the function f to be:

$f(x) = \dfrac{x^n(a-bx)^n}{n!}$

where $\pi = \frac{a}{b}$ and $n!$ is "n factorial," $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$.

Finally, a bit of notation: Well use the notation $f^{(j)}(x)$ to denote the j^{th} derivative of $f(x)$, so $f^{(2)}(x) = f''(x)$ and $f^{(17)}(x)$ would be the 17^{th} derivative of $f(x)$.

We're going to start by observing a few facts about f:

- **Fact 1:** $0 \leq f(x) \leq \dfrac{\pi^n a^n}{n!}$ if $0 \leq x \leq \pi$.
- **Fact 2:** f is a polynomial with integer coefficients, except for the factor of $\dfrac{1}{n!}$.
- **Fact 3:** $f(x) = f(\pi - x)$.
- **Fact 4:** For any derivative $f^{(j)}(x)$, we have that $f^{(j)}(x) = (-1)^j f^{(j)}(\pi - x)$. So in particular, $f^{(j)}(0) = (-1)^j f^{(j)}(\pi)$.
- **Fact 5:** If $j < n$, $f^{(j)}(0) = 0$.
- **Fact 6:** If $n \leq j \leq 2n$, then $f^{(j)}(0)$ is an integer.
- **Fact 7:** If $j < n$, $f^{(j)}(\pi) = 0$ and if $n \leq j \leq 2n$, $f^{(j)}(\pi)$ is an integer.
- **Fact 8:** If $j > 2n$, $f^{(j)}(x) = 0$.

Now we take a break from the facts for a moment to define a new function $g(x) = f(x) - f''(x) + f^{(4)}(x) - f^{(6)}(x) + \cdots + (-1)^j f^{(2j)}(x) + \cdots + (-1)^n f^{(2n)}(x)$. Back to the facts:

- **Fact 9:** $g(0)$ and $g(\pi)$ are integers.
- **Fact 10:** $g(x) + g''(x) = f(x)$.
- **Fact 11:** $\frac{d}{dx}[g'(x)\sin(x) - g(x)\cos(x)] = f(x)\sin(x)$.
- **Fact 12:** $\displaystyle\int_0^\pi f(x)\sin(x)\ dx = g(0) + g(\pi)$.

Putting Facts 9 and 12 together, we see that $\displaystyle\int_0^\pi f(x)\sin(x)\ dx$ is an integer. We'll work as a group to put all of this together to find a contradiction.