Exploring Normal Subgroups with GAP

Recall that the **left** and **right cosets** of a subgroup N in G are defined to be $aN = \{a*n \mid n \in N\}$ and $Na = \{n*a \mid n \in N\}$, respectively, where $a \in G$.

A subgroup N of a group G is said to be **normal** if $xN = Nx$ for all $x \in G$, or equivalently, if $xNx^{-1} = N$ for all $x \in G$.

If N is a normal subgroup of G, the cosets (left and right are the same!) form a group G/N called the **quotient group** under the operation $aN*bN = (a*b)N$.

Hopefully you now know enough about **gap** to be able to do this lab on your own, but don't hesitate to ask for help. You're here to learn the mathematics, not to become an expert **gap** jockey!

You'll need to get the function "CosetTable" loaded into **gap**. To do this, type in the following lines of code. (Notice the absence of a semi-colon at the end of some lines!)

```
gap> CosetTable : = function(g, n)
   > local x, y, tmp;
   > tmp := Flat(List(LeftCosets(g, n), x-> Elements(x)));
   > PrintArray(List(tmp, x->List(tmp, y->
   > Position(LeftCosets(g, n), x*y*n)))); end;
```

Question 1. Define the cyclic group of order 6 using

```
gap> r := (1, 2, 3, 4, 5, 6);
gap> G := Group(r);
```

and run the function CosetTable on G and the subgroup N generated by r^3:

```
gap> CosetTable(G, Subgroup(G, [r^3]));
```

a.) Explain why it's obvious that the subgroup generated by r^3 is normal in G.

b.) List the cosets of N.

c.) Pick two different non-identity cosets of N and verify that the product of your two cosets is a well-defined coset, i.e., that the product of any two representatives of these cosets is always in the same coset. (Note: Knowing that this group is abelian cuts your work in half!)

d.) Predict and then verify the results of the following two commands:

```
gap> CosetTable(g, Subgroup(G, [r]));
gap> CosetTable(g, Subgroup(G, [r^6]));
```

Question 2. Define G to be the symmetric group on 3 letters:

gap> G := SymmetricGroup(3);

Run the function CosetTable on the subgroup N generated by the permutation r := (1, 2, 3);
List the cosets and use your output to identify the structure of the quotient group of G by N.

Question 3. Keep G as defined above and repeat the previous step using r := (1, 2).

a.) Explain how your output differs from what you saw in the previous step.

b.) Use your output to identify two left cosets the product of which is **not** a left coset.

c.) Use your output to identify two left cosets the product of which **is** a left coset!

Question 4. Use r : = (1, 2, 3, 4, 5, 6); and rf := (2, 6)(3, 5); to define G as the dihedral group of order 12.

a.) Run the function CosetTable using the subgroup N generated by rf. Based on the output, is this subgroup normal or not?

b.) Run the function CosetTable for the subgroups $\langle rf^2 \rangle$, $\langle r^2 \rangle$, $\langle r \rangle$, and $\langle r, rf \rangle$. Describe what you see. The subgroups here are part of a composition series for G. Composition Series can be used to describe the structure of a group, by viewing subgroups as "smaller **simple** pieces of the whole". We will learn more about these later.

Question 5. (Extra Credit) Find another composition series for the group above. Take your data from either or both compositions, get your crayons, markers, watercolors, etc, to create an artistic representation which illustrates the quotient groups. Note: By choosing your colors carefully you can represent several quotient groups in one illustration.