Hartlaub - Fall 2009

Math 336 - Probability - Exam #2 Solutions

#1 Rewrite $h(t) = E[t^X] = .2t^{-2} + .1t^{-1} + .4t^0 + .1t + .2t^2$

a. The probability distribution of X is:

x	-2	-1	0	1	2
$P(X=x)$	$.2$	$.1$	$.4$	$.1$	$.2$

b. $E[X] = \frac{d}{dt} h(t)\Big|_{t=1} = -.4t^{-3} - .1t^{-2} + .1 + .4t\Big|_{t=1} = 0$

The distribution of X is symmetric about 0.

c. $E[X(X-1)] = \frac{d^2}{dt^2} h(t)\Big|_{t=1} = 1.2t^{-4} + .2t^{-3} + .4\Big|_{t=1}$

$= 1.2 + .2 + .4 = 1.8$

Thus, $Var(X) = E[X(X-1)] + E[X] - \{E[X]\}^2$

$= 1.8 + 0 - 0^2 = 1.8.$

#2. a. The ~~pdf~~ p.d.f. of $Y_{(1)}$ is $f_1(y) = \frac{n!}{(n-1)!}[1-F(y)]^{n-1} f(y)$

$F(y) = \int_\theta^y f(y)dy = \int_\theta^y e^{-(u-\theta)} du = -e^{-(u-\theta)}\Big|_\theta^y = 1 - e^{-(y-\theta)}$

Thus, $f_1(y) = n\left[e^{-(y-\theta)}\right]^{n-1} e^{-(y-\theta)}$

$= \begin{cases} n e^{-n(y-\theta)}, & y > \theta \\ 0, & \text{elsewhere} \end{cases}$

b. $E[Y_{(1)}] = \int_\theta^\infty y\, n e^{-n(y-\theta)} dy$

$\underset{\text{Let } z = y-\theta}{=} \int_0^\infty n(z+\theta) e^{-nz} dz$

$\underset{\text{Complete the Gamma distributions}}{=} n\Gamma(2)\left(\frac{1}{n}\right)^2 + n\theta\left(\frac{1}{n}\right) = \frac{1}{n} + \theta.$

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#3 a. $E[3Y_1 + 4Y_2 - 6Y_3] = 3(2) + 4(-1) - 6(4) = -22$

$Var[3Y_1 + 4Y_2 - 6Y_3] = 9(4) + 16(6) + 36(8)$
$+ 2(3)(4)(1) + 2(3)(-6)(-1) + 2(4)(-6)(0)$
$= 480$

#4. a. Integrating the joint density function over the region indicated under the restriction that $y_1 \le y_2$, we have

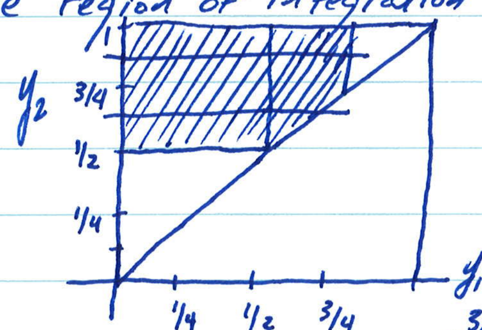
$$\int_0^1 \int_0^{y_2} K(1-y_2)\, dy_1\, dy_2 = \int_0^1 K(y_2 - y_2^2)\, dy_2$$

$$= K \left. \frac{y_2^2}{2} - \frac{y_2^3}{3} \right|_{y_2=0}^{1} = K\left(\frac{1}{2} - \frac{1}{3}\right)$$

Therefore, $K\left(\frac{1}{2} - \frac{1}{3}\right) = 1$ or $K\left(\frac{1}{6}\right) = 1$

so, $\boxed{K = 6}$

b. The region of integration is shown below



$$P\left(Y_1 \le \tfrac{3}{4}, Y_2 \ge \tfrac{1}{2}\right) = \int_{1/2}^{3/4} \int_0^{y_2} 6(1-y_2)\, dy_1\, dy_2 + \int_{3/4}^{1} \int_0^{3/4} 6(1-y_2)\, dy_1\, dy_2$$

$$= \int_{1/2}^{3/4} 6(y_2 - y_2^2)\, dy_2 + \int_{3/4}^{1} \frac{9}{2}(1-y_2)\, dy_2$$

$$= \left. 3y_2^2 - 2y_2^3 \right|_{y_2=1/2}^{3/4} + \left. \frac{9}{2}\left(y_2 - \frac{y_2^2}{2}\right) \right|_{y_2=3/4}^{1}$$

$$= \frac{22}{64} + \frac{9}{64} = \frac{31}{64} = .4844$$

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#4. c. $E[Y_1] = \int_0^1 \int_{y_1}^1 y_1 6(1-y_2)\,dy_2\,dy_1 = \int_0^1 y_1(3-6y_1+3y_1^2)\,dy_1$

$= \int_0^1 3y_1 - 6y_1^2 + 3y_1^3\,dy_1 = 1/4$

$E[Y_2] = \int_0^1 \int_0^{y_2} y_2\, 6(1-y_2)\,dy_1\,dy_2 = \int_0^1 y_2\, 6(y_2 - y_2^2)\,dy_2$

$= \int_0^1 6y_2^2 - 6y_2^3\,dy_2 = 1/2$

$E[Y_1 Y_2] = \int_0^1 \int_0^{y_2} 6y_1 y_2(1-y_2)\,dy_1\,dy_2 = \int_0^1 3(y_2^3 - y_2^4)\,dy_2$

$= \frac{3}{4} - \frac{3}{5} = \frac{3}{20}$

Thus, $Cov(Y_1, Y_2) = \frac{3}{20} - \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = \frac{3}{20} - \frac{1}{8} = \frac{1}{40}$

Since $Cov(Y_1, Y_2) \neq 0$, Y_1 and Y_2 are not independent.

#5. a. Taking the derivative we have

$f(y) = \frac{\alpha y^{\alpha-1}}{\theta^\alpha} \quad 0 \le y \le \theta$

b. We need $F_Y(y) = u = \left(\frac{y}{\theta}\right)^\alpha$ or $y = \theta u^{1/\alpha}$

Thus, $G(u) = \theta u^{1/\alpha}$ (i.e. Find the inverse c.d.f.)

#6. a. If $\lambda > 0$, we have

$P(Y = y \mid \lambda) = \frac{\lambda^y e^{-\lambda}}{y!}, \; y = 0, 1, 2, \ldots$

and $f(\lambda) = e^{-\lambda}, \; 0 \le \lambda$

Thus, $P(y) = \int_0^\infty P(Y=y, \lambda)\,d\lambda = \int_0^\infty P(Y=y \mid \lambda) f(\lambda)\,d\lambda$

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#6 a. cont. $$p(y) = \int_0^\infty \frac{\lambda^y e^{-\lambda}}{y!} e^{-\lambda}\, d\lambda = \frac{1}{y!}\int_0^\infty \lambda^y e^{-2\lambda}\, d\lambda$$

$$= \frac{\Gamma(y+1)(1/2)^{y+1}}{y!}\int_0^\infty \frac{\lambda^y e^{-2\lambda}}{\Gamma(y+1)\left(\frac{1}{2}\right)^{y+1}}\, d\lambda$$

$$= \frac{\Gamma(y+1)\left(\frac{1}{2}\right)^{y+1}}{\Gamma(y+1)} = \left(1/2\right)^{y+1}$$

$$P(y) = \left(\frac{1}{2}\right)^{y+1}, \quad y = 0, 1, 2, \ldots$$

b. $$E[Y] = \sum_{y=0}^{\infty} y\left(\frac{1}{2}\right)^{y+1} = \sum_{y=1}^{\infty} y\left(\frac{1}{2}\right)^{y+1} = \frac{1}{2}\sum_{y=1}^{\infty} y\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{y-1}$$

We recognize the sum as the expected value of a geometric random variable with $p = 1/2$ (which happens to be 2).

Therefore, $E[Y] = \frac{1}{2}(2) = 1$.

c. $$Var(Y) = E[Var(Y/\lambda)] + Var[E(Y/\lambda)]$$
$$= E[\lambda] + Var(\lambda)$$

Since λ is an exponential R.V. with parameter 1, we know

$$E[\lambda] = 1 \text{ and } Var(\lambda) = 1$$

Thus, $Var(Y) = 1 + 1 = 2$.

d. $9 = E[Y] + 5.657\,\sigma_Y$, so it is unlikely that Y exceeds 9.