Math 336 Exam #2 Solutions

Fall 2007 Hartlaub

(1) a. $Cov(X,Y) = E[XY] - E[X]E[Y]$

$E[XY] = 0 + 0 + 1(1)(.1) + 1 \cdot 2(.2) + 2(1)(.3) + 2 \cdot 2(.2) = 1.9$

$E[X] = 1(.6) + 2(.4) = 1.4$

$E[Y] = 0(.2) + 1(.3) + 2(.5) = 1.3$

$Cov(X,Y) = 1.9 - (1.4)(1.3) = \boxed{.08}$

b. $Var(X+Y) = Var(X) + Var(Y) + 2Cov(XY)$

$Var(X) = [1^2(.6) + 2^2(.4)] - 1.4^2 = .24$

$Var(Y) = [1^2(.3) + 2^2(.5)] - 1.3^2 = .61$

$\Rightarrow Var(X+Y) = .24 + .61 + 2(.08) = \boxed{1.01}$

c. $Cov(6X+4, Y-10) = 6\,Cov(X,Y) = 6(.08) = \boxed{.48}$

d. $P(X=1 \mid X+Y=3) = \dfrac{P(X=1, Y=2)}{P(X+Y=3)} = \dfrac{.3}{.3+.2} = \boxed{\dfrac{3}{5}}$

e. $M.A.D.(Y) = |0-1.3|.2 + |1-1.3|.3 + |2-1.3|.5$

$= .26 + .09 + .35$

$= \boxed{.70}$

f. Step 1: Find the conditional dist of $Y \mid X=1$

y	0	1	2
$f(y \mid X=1)$	2/6	1/6	3/6

Step 2: $E[Y \mid X=1] = 0(2/6) + 1(1/6) + 2(3/6) = 7/6$

$E[Y^2 \mid X=1] = 0^2(2/6) + 1^2(1/6) + 2^2(3/6) = 13/6$

$Var(Y \mid X=1) = \dfrac{13}{6} - \left(\dfrac{7}{6}\right)^2 = \dfrac{29}{36} = .8056$

$\sigma(Y \mid X=1) = \sqrt{.8056} = .8975$

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(2) $X \sim$ Geometric $(p = .3)$

a. $P(X=5) = (.7)^4(.3) = .0720$

b. $E[X] = \frac{1}{p} = \frac{1}{.3} = 3.3333$

c. $P(X=15 \mid X>10) = P(X=5) = .0720$ by memoryless property

(3) X = # of emissions in a one-second interval

$X \sim$ Poisson(λ) and $P(X=0) = .165$

a. $P(X=0) = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda} = .165$

$\Rightarrow -\lambda = \ln(.165)$

$\Rightarrow \boxed{\lambda = -\ln(.165) = 1.8018}$

b. $P(X=2) = \frac{e^{-1.8018}(1.8018)^2}{2!} = \boxed{.2678}$

c. two-second interval $\Rightarrow$ Poisson with parameter $M = 1.8018 + 1.8018$
$= 3.6036$

$P(Y=0) = \frac{e^{-3.6036}(3.6036)^0}{0!} = e^{-3.6036} = \boxed{.0272}$

d. four-second interval $\rightarrow$ Poisson with parameter $\lambda_4 = 4(1.8018)$
$= 7.2072$

Z = # of emissions in a 4 second interval.

$P(Z \le 2) = P(Z=0) + P(Z=1) + P(Z=2)$

$= \frac{e^{-7.2072}(7.2072)^0}{0!} + \frac{e^{-7.2072}(7.2072)^1}{1!} + \frac{e^{-7.2072}(7.2072)^2}{2!}$

$= .0007 + .0053 + .0193$

$= .0253$

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(4) $P(Y \text{ is odd}) = \sum_{y=1}^{\infty} \left(\frac{1}{2}\right)^{2y-1}$

$= \left(\frac{1}{2}\right)^{-1} \sum_{y=1}^{\infty} \left(\frac{1}{2}\right)^{2y}$

$= 2 \sum_{y=1}^{\infty} \left(\frac{1}{4}\right)^{y}$

$= 2 \left[\sum_{y=0}^{\infty} \left(\frac{1}{4}\right)^{y} - 1\right]$

$= 2 \left[\frac{1}{1-1/4} - 1\right]$

Geometric series

$= 2\left[\frac{4}{3} - 1\right] = 2\left[\frac{1}{3}\right] = \boxed{\frac{2}{3}}$

(5) Let W = # of trials necessary to obtain the 5^{th} success

$W \sim \text{Negbin}(5, .75)$

$P(W=8) = \binom{8-1}{5-1}(.75)^5(.25)^{8-5} = \binom{7}{4}(.75)^5(.25)^3$

$\boxed{P(W=8) = .1298}$

(6) $\eta(t) = E[t^X] = .2t + .2t^2 + .2t^3 + .4t^4$

(a) $E[t^X] = \sum_x t^x P(X=x)$

Since $P(X=x)$ is the coefficient of t^x, the prob dist of X is

x	1	2	3	4
$P(X=x)$	.2	.2	.2	.4

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6 b $\quad 2^{nd}$ factorial moment $= h''(1)$

$$h'(t) = .2 + .4t + .6t^2 + 1.6t^3$$

$$h''(t) = .4 + 1.2t + 4.8t^2$$

$$h''(1) = .4 + 1.2 + 4.8 = \boxed{6.4 = E[X(X-1)]}$$

7 a. $$F(y) = \int_{-1}^{y} \frac{3}{2}u^2\,du = \frac{1}{2}u^3\Big|_{-1}^{y} = \begin{cases} 0, & \text{for } y < -1 \\ \frac{1}{2}(y^3+1), & \text{for } -1 \le y < 1 \\ 1, & y \ge 1 \end{cases}$$

b. $$F_U(u) = P(U \le u) = P(3Y \le u)$$

$$= P\left(Y \le \frac{u}{3}\right)$$

$$= F_Y\left(\frac{u}{3}\right)$$

$$= \frac{1}{2}\left(\frac{u^3}{27} + 1\right) \quad \text{for } -3 \le u \le 3$$

Support

$$-1 \le Y \le 1 \xrightarrow{U=3Y} -3 \le U \le 3$$

Differentiating with respect to u we have

$$f_U(u) = \frac{d}{du}F_U(u) = \frac{d}{du}\left(\frac{1}{2}\left(\frac{u^3}{27}+1\right)\right)$$

$$= \begin{cases} \frac{u^2}{18}, & -3 \le u \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

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Extra Credit. (a) The joint p.f. of (X_i, Y_i) is as follows:

$$f(1,0) = f(-1,0) = f(0,1) = f(0,-1) = 1/4$$

The marginal distribution of X_i is:

x_i	-1	0	1
$P(X_i = x_i)$	$1/4$	$1/2$	$1/4$

Similarly, the marginal distribution of Y_i is

y_i	-1	0	1
$P(Y_i = y_i)$	$1/4$	$1/2$	$1/4$

Since $f(1,0) = 1/4 \neq \frac{1}{8} = (1/4)(1/2) = P(X_i = 1)P(Y_i = 0)$, we conclude that X_i and Y_i are not independent.

(b) Let $U_n = \sum_{i=1}^{n} X_i$ = # of blocks you are North of the starting point after n moves

and $V_n = \sum_{i=1}^{n} Y_i$ = " " " " East " " " " " "

The distance from the starting point after n steps is

$$\sqrt{E[U_n^2 + V_n^2]} = \sqrt{E[U_n^2] + E[V_n^2]} = \sqrt{\frac{n}{2} + \frac{n}{2}} = \sqrt{n}$$

Note that $E[U_n] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} 0 = 0 = E[V_n]$

and $Var[U_n] = Var[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} Var[X_i] = \sum_{i=1}^{n} \frac{1}{2} = \frac{n}{2} = Var(V_n)$