Math 336 - Exam #2 Solutions

Fall 2005 - Hartlaub

#1. $h(t) = E[t^Y] = \sum_{y=0}^{n} t^y \binom{n}{y} p^y (1-p)^{n-y}$

$= \sum_{y=0}^{n} \binom{n}{y} (pt)^y (1-p)^{n-y}$

$= (pt + (1-p))^n$

Recall $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

$h'(t) = \frac{d}{dt} (pt + (1-p))^n = n(pt+(1-p))^{n-1} \cdot p$

$= np(pt+(1-p))^{n-1}$

$h'(1) = E[Y] = np(p+(1-p))^n = np$ ✓

#2. $\mu = .5$, $\sigma = .01$

Thus, the interval from .48 to .52 contains values that are within 2 standard deviations. Chebychev's inequality says

$$P(|X - E[X]| \le k\sigma) \ge 1 - \frac{1}{k^2}$$

Therefore, the lower bound for $k=2$ is

$$P(|X - E[X]| \le 2\sigma) \ge 1 - \frac{1}{2^2} = \frac{3}{4}$$

According to Chebychev's inequality, we would expect $\frac{3}{4}(400) = 300$ of the coins to have a diameter between .48 and .52

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#3. $X \sim \text{Poisson}(7)$

a.) $P(X \geq 2) = 1 - P(X < 2) = 1 - P(X \leq 1)$

$$= 1 - \frac{7^0 e^{-7}}{0!} - \frac{7^1 e^{-7}}{1!}$$

$$= 1 - e^{-7} - 7e^{-7} = .9927$$

b.) Let Y = total # of customers arriving in a 2 hour Period.
Using properties of a Poisson process (which were discussed in class), $Y \sim \text{Poisson}(\lambda = 2 \times 7 = 14)$

$$P(Y=2) = \frac{14^2 e^{-14}}{2!} = .00008149$$

c.) The probability, $P(Y=2)$, would stay the same!
Justification: Let X_1 = # of customers arriving between 1:00 and 2:00
and ~~with~~ X_2 = " " " " " 3:00 and 4:00

$$P(Y=2) = P(X_1 + X_2 = 2)$$

$$= P(X_1 = 0, X_2 = 2) + P(X_1 = 1, X_2 = 1) + P(X_1 = 2, X_2 = 0)$$

same

$$= 2\,P(X_1 = 0, X_2 = 2) + P(X_1 = 1, X_2 = 1)$$

$$= 2\left(e^{-7} \times \frac{7^2 e^{-7}}{2!}\right) + \left(\frac{7^1 e^{-7}}{1!}\right)\left(\frac{7^1 e^{-7}}{1!}\right)$$

$$= 7^2 e^{-14} + 7^2 e^{-14}$$

$$= 2\, 7^2 e^{-14}$$

$$= .000081490 \checkmark$$

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#4. $f(y) = \begin{cases} cy^2 + y, & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$

a. $\int_{-\infty}^{\infty} (cy^2 + y)\,dy = \int_0^1 cy^2 + y\,dy$

$$= c\frac{y^3}{3} + \frac{y^2}{2} \Big|_{y=0}^{1}$$

$$= \frac{c}{3} + \frac{1}{2}$$

Setting $\frac{c}{3} + \frac{1}{2} = 1$ yields $\frac{c}{3} = \frac{1}{2}$ or $\boxed{c = \frac{3}{2}}$

b. $F(y) = \int_0^y \frac{3}{2}u^2 + u\,du = \frac{3}{2}\frac{(u^3)}{3} + \frac{u^2}{2}\Big|_{u=0}^{y}$

$$= \frac{y^3}{2} + \frac{y^2}{2}$$

Thus $F(y) = \begin{cases} 0 & y \le 0 \\ y^3/2 + y^2/2 & 0 \le y \le 1 \\ 1 & y > 1 \end{cases}$

c. $P(Y \le \underbrace{\tfrac{1}{2}}_{\text{convert 30 min to hours}}) = F(\frac{1}{2}) = \frac{(1/2)^3}{2} + \frac{(1/2)^2}{2} =$

$$= \frac{1}{16} + \frac{1}{8} = \boxed{\frac{3}{16}} = .1875$$

d. $P(Y \ge \frac{1}{2} \mid Y \ge 1/4) = \frac{P(Y \ge 1/2)}{P(Y \ge 1/4)} = \frac{1 - F(1/2)}{1 - F(1/4)}$

Again, convert to hours

$$= \frac{13/16}{1 - \left[\frac{1}{128} + \frac{1}{32}\right]} = \frac{13/16}{123/128} = \boxed{\cancel{\frac{104}{128} = .8125}}$$

$\frac{104}{123} = \frac{1664}{1968} = .845528$

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#5. Let R = the radius of the crater.

$$E[R] = 10 \Rightarrow \frac{1}{\lambda} = 10 \Rightarrow \boxed{\lambda = \frac{1}{10}}$$

Thus, $R \sim \text{Exponential}(1/10)$

$E[R] = 1/(1/10) = 10$
$Var(R) = (1/(1/10))^2 = 10^2 = 100$

We want the expected area, where $A = \pi R^2$

$$E(A) = E[\pi R^2] = \pi E[R^2]$$
$$= \pi [Var(R) + \{E[R]\}^2]$$
$$= \pi [100 + 10^2]$$
$$= 200\pi.$$

#6. a. False
b. True
c. True
e. False.