Math 336 Fall 2002

Exam 2 Solutions

#1. $f(x) = \frac{1}{2} e^{-|x|}, \quad -\infty < x < \infty$

If $x \le 0$, then $F_X(x) = \int_{-\infty}^{x} \frac{1}{2} e^{-|y|} dy = \int_{-\infty}^{x} \frac{1}{2} e^{y} dy$

$$= \frac{1}{2} e^{y} \Big|_{-\infty}^{x}$$

$$= \frac{1}{2} e^{x}$$

If $x > 0$, then $F_X(x) = F_X(0) + P(0 < X \le x)$

$$= \frac{1}{2} + \int_0^x \frac{1}{2} e^{-|y|} dy$$

$$= \frac{1}{2} + \int_0^x \frac{1}{2} e^{-y} dy$$

$$= \frac{1}{2} + -\frac{1}{2} e^{-y} \Big|_0^x$$

$$= \frac{1}{2} + \left(-\frac{1}{2} e^{-x} - -\frac{1}{2} e^{0} \right)$$

$$= 1 - \frac{1}{2} e^{-x}$$

$$F_X(x) = \begin{cases} \frac{1}{2} e^{x}, & x \le 0 \\ 1 - \frac{1}{2} e^{-x}, & x > 0 \end{cases}$$

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#2.

X	4	3	2	1	0
prob	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{9}{13}$

$$h_t(X) = E[t^X] = t^4 \cdot \frac{1}{13} + t^3 \cdot \frac{1}{13} + t^2 \cdot \frac{1}{13} + t^1 \frac{1}{13} + t^0 \frac{9}{13}$$
$$= \frac{9}{13} + \frac{t}{13} + \frac{t^2}{13} + \frac{t^3}{13} + \frac{t^4}{13}$$

$$E[X] = h_t'(1)$$

$$h_t'(X) = \frac{d}{dt} h_t(X) = \frac{d}{dt}\left(\frac{9}{13} + \frac{t}{13} + \frac{t^2}{13} + \frac{t^3}{13} + \frac{t^4}{13}\right)$$
$$= \frac{1}{13} + \frac{2t}{13} + \frac{3t^2}{13} + \frac{4t^3}{13}$$
$$h_t'(1) = \frac{1}{13} + \frac{2 \cdot 1}{13} + \frac{3 \cdot 1^2}{13} + \frac{4 \cdot 1^3}{13} = \frac{10}{13}$$

#3 Let Y_1 = # of customers of type 1 $\quad Y_1 \sim Poi(3)$
Y_2 = # of customers of type 2 $\quad Y_2 \sim Poi(5)$
Y_3 = # of customers of type 3 $\quad Y_3 \sim Poi(10)$

$X = Y_1 + Y_2 + Y_3$, so X is a poisson process with an average of $(3+5+10) = 18$ arrivals per hour.

The mean # of arrivals in a 5 minute period is $\lambda t = 18 \cdot \frac{5}{60} = 1.5$ so the number of arrivals of any type in a 5 minute period is a Poisson process with mean 1.5.

$$P(\text{no arrivals in a 5 min. period}) = \frac{e^{-1.5}\, 1.5^0}{0!} = e^{-1.5} = .2231$$

Poisson dist with $\mu = 1.5$

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#4. The joint distribution of X and Y is

		Y = 2	3	
X	1	4/10	4/15	4/6
	2	4/20	4/30	2/3

The conditional distribution of Y given $X=1$ is

y	2	3
$P(y \mid X=1)$	$\frac{4/10}{4/6} = 6/10 = 3/5 = .6$	$\frac{4/15}{4/6} = \frac{6}{15} = \frac{2}{5} = .4$

$$E[Y \mid X=1] = 2 \cdot \frac{3}{5} + 3 \cdot \frac{2}{5} = \frac{12}{5} = \boxed{2.4}$$

#5. Poisson Dist. $f(x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, 2, \ldots$

$$3P(X=1) = P(X=2) \Rightarrow 3 \cdot \frac{e^{-\lambda}\lambda^1}{1!} = \frac{e^{-\lambda}\lambda^2}{2!}$$

$$\Rightarrow 3e^{-\lambda}\lambda = \frac{e^{-\lambda}\lambda^2}{2}$$

$$\Rightarrow 6 = \frac{e^{-\lambda}\lambda^2}{e^{-\lambda}\lambda}$$

$$\Rightarrow \boxed{\lambda = 6}$$

Thus, $P(X=4) = \dfrac{e^{-6}6^4}{4!} = .1339$

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#6. $\int_0^4 c\sqrt{x}\,dx = 1 \iff \int_0^4 c x^{1/2} dx = 1$

$$\iff c \left.\frac{x^{3/2}}{3/2}\right|_{x=0}^{4} = 1$$

$$\iff \frac{2c}{3} \left. x^{3/2}\right|_{x=0}^{4} = 1$$

$$\iff \frac{2c}{3} 4^{3/2} = 1$$

$$\iff \frac{2c}{3} \cdot 8 = 1$$

$$\iff \boxed{c = \frac{3}{16}}$$

#7.

Sample Space	Prob.	Z	W
HH	$(.4)^2 = .16$	1	2
HT	$(.4)(.6) = .24$	1	1
TH	$(.6)(.4) = .24$	0	1
TT	$(.6)^2 = .36$	0	0

The joint dist of Z and W is

Z \ W	0	1	2	
0	.36	.24	0	.6
1	0	.24	.16	.4
	.36	.48	.16	1

$$Cov(Z,W) = E[ZW] - E[Z]E[W]$$

where $E[ZW] = 1\cdot1\cdot(.24) + 1\cdot2\cdot(.16) = .56$

$E[Z] = 1\cdot(.4) = .4$

$E[W] = 1\cdot(.48) + 2(.16) = .80$

Thus, $Cov(Z,W) = .56 - (.8)(.4) = .24$