Probability - Math 336

Exam 1 Solutions - Fall 2007

#1. Let N = # of floors at which the elevator makes a stop to let out one or more people.

N is a counting variable which can be written as the sum of ten indicator variables (just like Binomial R.V.'s)

$$N = \sum_{i=1}^{10} I_{(\text{at least one person chooses floor } i)}$$

$$E[N] = E\left[\sum_{i=1}^{10} I_{(F_i)}\right], \text{ where } F_i = \{\text{at least one person chooses floor } i\}$$

$$= \sum_{i=1}^{10} E\left[I_{(F_i)}\right]$$

$$= \sum_{i=1}^{10} P(\text{at least one person chooses floor } i)$$

Now, for each i

$$P(F_i) = 1 - P(\text{nobody choses floor } i)$$

$$= 1 - (9/10)^{12} \leftarrow \text{by indep of people's choices.}$$

$$= 0.7176$$

$$E[N] = \sum_{i=1}^{10}\left[1-(9/10)^{12}\right] = 10\left[1-(9/10)^{12}\right] = 7.176$$

#2. a.

Number of cookies in each bowl

	Bowl 1	Bowl 2	Total
Choc. Chip	10	20	30
Plain	30	20	50
	40	40	80

Relative Frequency of cookies in each bowl

	Bowl 1	Bowl 2	Totals
Choc. Chip	.125	.250	.375
Plain	.375	.250	.625
	.500	.500	1.00

$$P(B1 \mid Plain) = \frac{P(Plain \mid B1)P(B1)}{P(Plain \mid B1)P(B1) + P(Plain \mid B2)P(B2)}$$

$$= \frac{.75(.5)}{.75(.5) + .5(.5)} = \frac{.375}{.625} = .60$$

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#2 (b)

$$P(BC/pos) \underset{\text{Bayes Thm.}}{=} \frac{P(pos/BC)P(BC)}{P(pos/BC)P(BC)+P(pos/BC^c)P(BC^c)}$$

$$= \frac{(0.8)(0.01)}{(0.8)(0.01)+(0.096)(0.99)}$$

$$= \frac{0.008}{0.008+0.095} = \frac{0.008}{0.103} = \boxed{0.0777}$$

OR $\boxed{7.8\%}$

#3. Let $H = \{\#\text{ of hearts in a 5 card hand}\}$

$$P(H=2) \underset{\text{Hypergeometric dist}}{=} \frac{\binom{13}{2}\binom{52-13}{5-2}}{\binom{52}{5}} = \frac{\binom{13}{2}\binom{39}{3}}{\binom{52}{5}} = \frac{78 \times 9{,}139}{2{,}598{,}960}$$

$$P(H=2) = .27428$$

#4. Let X = # of male geckos hatched.

$X \sim b(n=20, p=.35)$

(a) $E[X] = np = 20(.35) = 7$

(b) $P(X \geq 5) = 1 - P(X<5) = 1 - P(X \leq 4) \underset{\text{MTB}}{=} 1 - .1182 = .8818$

(c) $P(X=7) = \binom{20}{7}(.35)^7(1-.35)^{20-7} \underset{\text{MTB}}{=} 0.1844$

(d) Use geometric dist or $P(F_1F_2F_3F_4M_5) = .65^4(.35) = .0625$

(e) If $n=2000$, then $npq = 2000(.35)(.65) = 455 > 5$. Thus, the Normal approximation would be appropriate. The Poisson approximation is useful when p is close to 1 (or close to 0). No it should not be used in this situation.

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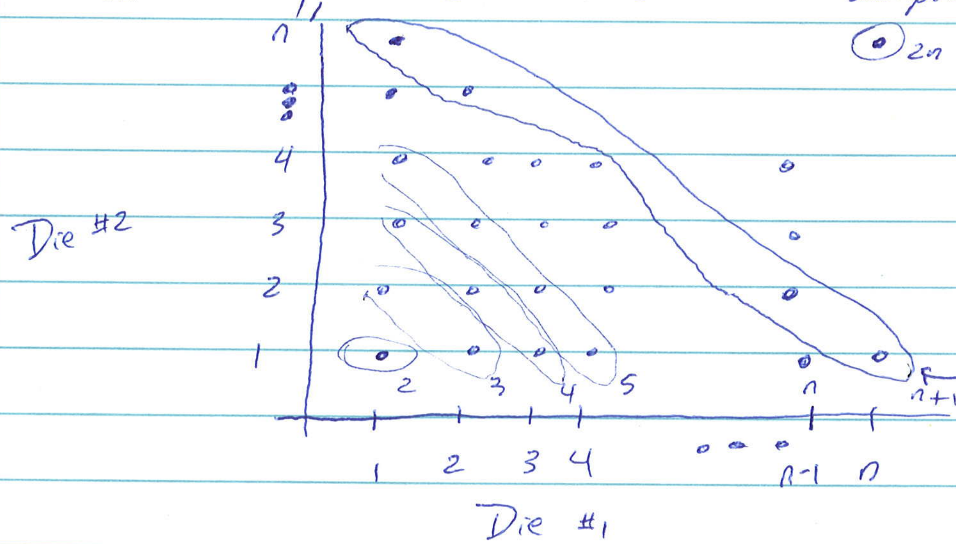
#5. a. Use multinomial dist with 6 categories and $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = \frac{1}{6}$.

$$P(6 \text{ faces in } 6 \text{ rolls}) = \frac{6!}{1!1!1!1!1!1!}\left(\frac{1}{6}\right)^6 = \frac{5}{324} = .0154$$

b. $$P(4 \text{ 1's and } 2 \text{ 2's}) = \frac{6!}{4!2!0!0!0!0!}\left(\frac{1}{6}\right)^4\left(\frac{1}{6}\right)^2 = \frac{5}{15{,}552} = .0003$$

#6. a. As we have seen with previous dice problems, the distribution of the sum is symmetric. The center of the distribution is the most likely value. Thus, the most probable sum is $E[X_1 + X_2] = \frac{n+1}{2} + \frac{n+1}{2} = n+1$.

Alternatively, we could consider the sample space:



$P(X_1+X_2=2) = P(X_1+X_2=2n) = \frac{1}{n^2}$

$P(X_1+X_2=3) = P(X_1+X_2=2n-1) = \frac{2}{n^2}$

$P(X_1+X_2=4) = P(X_1+X_2=2n-2) = \frac{3}{n^2}$

$\vdots$

$P(X_1+X_2=n+1) = \frac{n}{n^2} = \frac{1}{n}$

Most probable value $= (n+1)$

b.

X	1	2	3	...	n
prob	$1/n$	$1/n$	$1/n$		$1/n$

or

$$f_X(x) = \begin{cases} 1/n, & x = 1, 2, \ldots, n \\ 0, & \text{otherwise.} \end{cases}$$

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#6 c. Consider the case where $n=1$:

$$E[X^2] = \frac{1}{1}(1^2) = 1 = \frac{(1+1)(2(1)+1)}{6} = \frac{2(3)}{6} \checkmark$$

Induction hypothesis: $\overset{(n=k)}{E[X^2]} = \frac{(k+1)(2k+1)}{6}$

Prove the relationship holds for $n=k+1$:

$$E[X^2] = 1^2 \cdot \frac{1}{k+1} + 2^2\frac{1}{k+1} + \cdots + k^2\frac{1}{k+1} + (k+1)^2\frac{1}{k+1}$$

$$= \frac{1}{k+1}\left(1^2 + 2^2 + \cdots + k^2 + (k+1)^2\right)$$

$$= \frac{k}{k+1}\left(1^2\frac{1}{k} + 2^2\frac{1}{k} + \cdots + k^2\frac{1}{k}\right) + k+1$$

$$\underset{\text{induction hypothesis}}{=} \frac{k}{\cancel{k+1}}\left(\frac{\cancel{(k+1)}(2k+1)}{6}\right) + k+1$$

$$= \frac{k(2k+1)}{6} + \frac{6(k+1)}{6}$$

$$= \frac{2k^2 + k + 6k + 6}{6}$$

$$= \frac{2k^2 + 7k + 6}{6}$$

$$= \frac{(k+2)(2k+3)}{6}$$

$$= \frac{((k+1)+1)(2(k+1)+1)}{6} \checkmark$$