Probability in Options Pricing

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December 14, 2012

What is an Option?

Let S_0 be the current price of a share of stock. Considering the following options with that reach maturity at time T.

Call Option

At time T, the holder of a call option has the right, but not the obligation, to buy one share of stock at a previously agreed upon price, known as the strike price, and often denoted K. Clearly, this option is only profitable if $S_T > K$.

Put Option

At time T, the holder of a put option has the right, but not the obligation, to sell one share of stock at the strike price K. Clearly, this option is only profitable if $S_T < K$.

Writers of options charge a premium of P_0 and C_0 for put and call options, respectively. *How do we determine the fair price of put and call options?*

Random Walks

- Probability theory used in everything from physics to biology
- Describes the iterated movements of an object with steps in a random directions.
- Simple features may include discrete step distance, identical step magnitude, discrete time steps or independent steps.
- Complicated models have more freedom in direction, different step sizes, time delays.

Random Walks

- Let Z_k for $k \geq 1$ be independently, identically distributed. Then $S_n = \sum_{k=1}^{n} Z_k$ for $n \in \mathbb{N}$ is a random walk.
- Stationary, so $Z_k = S_k - S_{k-1}$ have identical distributions.
- $S_{m+n} - S_n$ is a step with m time units.
- Infinite divisibility.
- As the time goes to 0, we can get a closer look at the process, and use the central limit theorem.
- $\frac{S_n - E[S_n]}{\sigma\sqrt{n}} \xrightarrow{d} Z \stackrel{d}{=} N(0,1)$, $X_t \stackrel{d}{=} N(0,t)$.

Brownian Motion

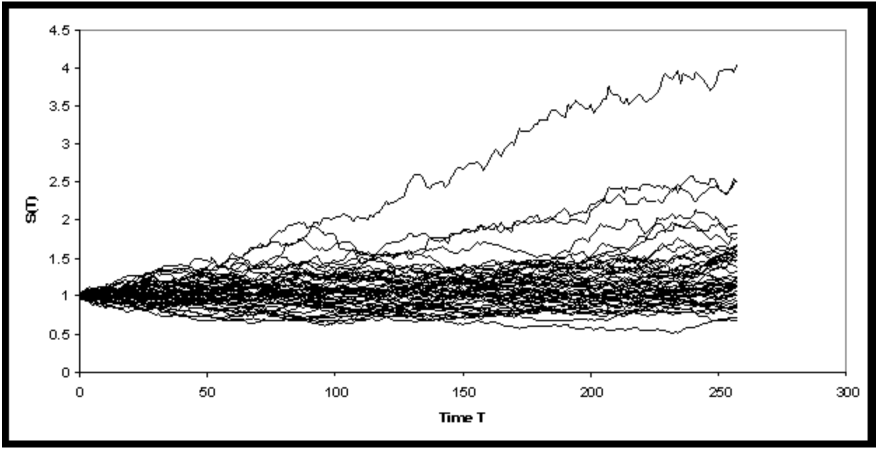
- The term initially described the random movements of particles as a result of collisions with smaller objects such as atoms and molecules.
- Here is an example of two dimensional Brownian motion.
- It is a continuous-time stochastic process, meaning that it is the limit of a random walk as the length of the time intervals approach zero.
- At one point in time, Brownian motion was used to describe the stock market, but this description was flawed, as Brownian motion would allow stock prices to drop below zero, when we know this is not possible.
- How do we augment the idea of Brownian motion to arrive at a model that we can use to describe the path of a stock price over time?

Geometric Brownian Motion

- Define $S_t = S_0 e^{X_t}$ where $X_t = \sigma B(t) + \mu t$ is a Brownian motion equation with drift parameter μ and diffusion parameter σ. The drift parameter is the average increase in S_t and the the diffusion parameter is a measure of volatility.[1]
- Since our Brownian motion term X_t is normally distributed with mean $\mu t + \ln(S_0)$ and variance $\sigma^2 t$ by assumption, we conclude that S_t follows the lognormal distribution.[2]
- Geometric Brownian Motion can also be represented by the stochastic differential equation $dS_t = \mu S_t dt + \sigma S_t dX_t$.

[1] [3], p. 1
[2] [3], p. 1

Geometric Brownian Motion



This is a plot of 50 possible outcomes of Geometric Brownian motion with $\mu = 0.001$, $\sigma = 0.02$, and intial stock price of $S_0 = 1$.

Black-Scholes Derivation

- Up until now, we have discussed how stock prices may be described by geometric brownian motion, and how geometric brownian motion may be modeled using the lognormal distribution. How can we use this knowledge to our advantage?
- Using some knowledge from portfolio allocation, if we assume that investors are risk neutral, then the price of the call will be equal to the time discounted expected payoff. Recall that if $K > S_T$, then the call will yield zero payoff. Then, it follows that the probability weighted average of the payoffs is equal to the probability that $S_T > K$ times the profit earned on the option. Algebraically, this can be expressed as $(E[S_T|S_T > K] - K) \cdot P\{S_T > K\}$.[3]
- By properties of conditional expectation, this is:

$$[\frac{\int_K^\infty S_T f(S_T) dS_T}{\int_K^\infty f(S_T) dS_T} - K] \cdot \int_K^\infty f(S_T) dS_T$$

From this, we can simplify to:

$$\int_K^\infty S_T f(S_T) dS_T - K \cdot \int_K^\infty f(S_T) dS_T$$

[3] This derivation follows the work of Professor Melick.

Black-Scholes Derivation (Using the Lognormal)

- Now, given that S_T follows the lognormal distribution, we know can rewrite our previous equation as follows:

$$\int_K^\infty S_T \frac{1}{S_T \sigma \sqrt{2\pi}\sqrt{t}} e^{\frac{-(\ln(S_T)-\mu)^2}{2\sigma^2 t}} dS_T - K \cdot \int_K^\infty \frac{1}{S_T \sigma \sqrt{2\pi}\sqrt{t}} e^{\frac{-(\ln(S_T)-\mu)^2}{2\sigma^2 t}} dS_T$$

- Next, we cancel the S_T from numerator and denominator of the first integral, and use the complement rule to change the bounds of integration on the second integral to get:

$$\int_K^\infty \frac{1}{\sigma \sqrt{2\pi}\sqrt{t}} e^{\frac{-(\ln(S_T)-\mu)^2}{2\sigma^2 t}} dS_T - K \cdot (1 - \int_0^K \frac{1}{S_T \sigma \sqrt{2\pi}\sqrt{t}} e^{\frac{-(\ln(S_T)-\mu)^2}{2\sigma^2 t}} dS_T)$$

Black-Scholes Derivation (Simplifying the Second Integral)

- In order to make some headway, let's narrow our focus down to the second integral, letting $z = \ln(S_T)$, and therefore $dz = \frac{1}{S_T} dS_T$. Clearly, this implies $dS_T = S_T dz$. Substituting this into the second integral yields:

$$\int_{-\infty}^{\ln(K)} \frac{1}{\sigma\sqrt{2\pi}\sqrt{t}} e^{\frac{-(z-\mu)^2}{2\sigma^2 t}} dz$$

- Taking a closer look, we can see that this is a form of the normal CDF. We normalize it to get:

$$\Phi(\frac{\ln(K) - \mu}{\sigma\sqrt{t}})$$

- Then, we can take this simplified form of the second integral, and substitute it back into our larger equation, which yields:

$$\int_{K}^{\infty} \frac{1}{\sigma\sqrt{2\pi}\sqrt{t}} e^{\frac{-(\ln(S_T)-\mu)^2}{2\sigma^2 t}} dS_T - K \cdot (1 - \Phi(\frac{\ln(K) - \mu}{\sigma\sqrt{t}}))$$

Black-Scholes Derivation (Simplifying the First Integral)

- Having simplified the second integral about as much as we can, let's turn our attention to the first integral. Let $z = \frac{\ln(S_T)-\mu}{\sigma\sqrt{t}}$. Then it follows that $\ln(S_T) = z\sigma\sqrt{t} + \mu$, so we are assured that $S_T = e^{z\sigma\sqrt{t}+\mu}$. Taking the derivate of z with respect to S_T, we get $\frac{dz}{dS_T} = \frac{1}{S_T\sigma\sqrt{t}}$, so we solve for dS_T to get:

$$dS_T = S_T\sigma\sqrt{t}dz = e^{z\sigma\sqrt{t}+\mu}\sigma\sqrt{t}dz$$

- Substituting in for z, out first integral becomes the following:

$$\int_{\frac{\ln(K)-\mu}{\sigma\sqrt{t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2+z\sigma\sqrt{t}+\mu} dz$$

- Let $B = \frac{1}{2}$ and $\gamma = -\sigma\sqrt{t}$. Then, we can substitute this values into our integral, and pull the constants out of the integral to get:

$$\frac{e^{\mu}}{\sqrt{2\pi}} \int_{\frac{\ln(K)-\mu}{\sigma\sqrt{t}}}^{\infty} e^{\frac{-z^2}{4B}-\gamma z} dz$$

Black-Scholes Derivation (The Tricky Part)

- From here, it may seem as if we have reached an impasse. However, it turns of that this integral can be transformed into something much easier to deal with, as seen below:

$$e^{\mu+\sigma^2 t/2}\Phi(\frac{\mu+\sigma^2 t-\ln(K)}{\sigma\sqrt{t}})$$

- Having simplified the first integral as much as we can, we substitute this simpler form into our earlier equation to get:

$$e^{\mu+\frac{\sigma^2 t}{2}}\cdot\Phi(\frac{\mu+\sigma^2 t-\ln(K)}{\sigma\sqrt{t}})-K\cdot\Phi(\frac{\mu-\ln(K)}{\sigma\sqrt{t}})$$

- Now that we have to probability weighted payoff, we adjust for time discount with continuous compounding to get the price of the call:

$$C=e^{-rt}(e^{\mu+\frac{\sigma^2 t}{2}}\cdot\Phi(\frac{\mu+\sigma^2 t-\ln(K)}{\sigma\sqrt{t}})-K\cdot\Phi(\frac{\mu-\ln(K)}{\sigma\sqrt{t}}))$$

Black-Scholes Derivation (Putting it All Together)

- By now, we have made quite a bit of progress. However, we can simplify even further to eliminate μ. To do so, recall that $E[S_T] = e^{\mu + \frac{\sigma^2 t}{2}}$. Furthermore, we know from porfolio allocation that if a stock pays no dividends $E[S_T] = S_0 e^{r \cdot t}$.
- Then, we set these two equations equal to eachother, and solve for μ to get $\mu = \ln(S_0) + rt - \frac{\sigma^2 t}{2}$.
- Substituting in for μ in our other equation, we get: $C = e^{-rt}(e^{(\ln(S_0)+rt-\frac{\sigma^2 t}{2})+\frac{\sigma^2 t}{2}} \cdot \Phi(\frac{(\ln(S_0)+rt-\frac{\sigma^2 t}{2})+\sigma^2 t-\ln(K)}{\sigma\sqrt{t}}) - K\Phi(\frac{(\ln(S_0)+rt-\frac{\sigma^2 t}{2})-\ln(K)}{\sigma\sqrt{t}}))$
- Finally, this simplifies to:

$$C = S_0\phi(d_1) - Ke^{-rt}\phi(d_2)$$

Where we define d_1 and d_2 such that $d_1 = \frac{\ln(S_0/K)+(r+\sigma^2/2)t}{\sigma\sqrt{t}}$ and $d_2 = d_1 - \sigma\sqrt{t}$.

How can we use Black-Scholes?

- All of the variables used in Black-Scholes are readily available, with the exception of σ. If we can find some measure of sigma, then we can determine whether options are correctly priced, and potentially profit from mispricings.
- Since the price of call options is readily available, we can solve for σ, the implied volatility of the stock. This can tell us all sorts of things, including the market's perception of the future potential of the stock.
- Black-Scholes is a very important mathematical model in the financial world. There are other incarnations that can be used to price other derivatives, including American options.

Concluding Remarks

- We started out with a discussion of random walks, realizing that the as the time interval between steps approaches zero, the process approaches Brownian Motion.
- We decided that Brownian motion does not accurately portray the price of stocks, as stock prices are neither continuous nor negative.
- We concluded that Geometric Brownian motion was the best alternative to Brownian motion, and it corrects most of out problems.
- By looking at how Geometric Brownian motion behaves, we were able to realize that S_t has a lognormal distribution.
- Using the pdf of the lognormal distribution and some conditional probability, we were able to derive the Black-Scholes equation for pricing European Options.
- Ultimately, with a good understanding of probability, we can derive one of the most influential models in the financial world.

References

- Cherubini, U., Della Lunga, G., Mulinacci, S., & Rossi, P. (2010). *Fourier transform methods in finance* (Vol. 464). Wiley.
- Ross, S. M. (2011). *An elementary introduction to mathematical finance.* Cambridge University Press.
- Sigman, K. (2006). *Geometric brownian motion.* Informally published manuscript, Columbia University.
- *www.vosesoftware.com/ModelRiskHelp/index.htm*