Math 224
Quiz 4
Thursday, October 18, 2007

Note: You are allowed to use Maple for computation on this quiz.

1. Find the volume of the 4-box in $\mathbf{R}^5$ determined by the vectors

$$[1, 1, 1, 0, 1],\ [0, 1, 1, 0, 0],\ [3, 0, 1, 0, 0],\ [1, -1, 0, 0, 1].$$

Solution. We form the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix}.$$

Using Maple, we compute $\det(A^T A) = 9$, so the volume is $\boxed{\sqrt{9} = 3}$.

2. Determine whether the points $(2, 0, 1, 3)$, $(3, 1, 0, 1)$, $(-1, 2, 0, 4)$, and $(3, 1, 2, 4)$ lie in a plane in $\mathbf{R}^4$.

Solution. The points are coplanar if and only if the 3-box in $\mathbf{R}^4$ determined by

$$\begin{aligned} (3, 1, 0, 1) - (2, 0, 1, 3) &= [1, 1, -1, -2] \\ (-1, 2, 0, 4) - (2, 0, 1, 3) &= [-3, 2, -1, 1] \\ (3, 1, 2, 4) - (2, 0, 1, 3) &= [1, 1, 1, 1] \end{aligned}$$

We construct the matrix A with these vectors as column vectors, and find that $\det(A^T A) = 378$, so the points are $\boxed{\text{not coplanar}}$.

3. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be defined by $T([x, y, z]) = [x - 2y, 3x + z, 4x + 3y]$. Find the volume of the image under T of the box $0 \le x \le 2$, $-1 \le y \le 3$, $2 \le z \le 5$ in $\mathbf{R}^3$.

Solution. The standard matrix representation for T is

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 0 & 1 \\ 4 & 3 & 0 \end{bmatrix}.$$

We compute $\det(A) = -11$, so the volume-change factor is 11. The original box in $\mathbf{R}^3$ has volume $(2 - 0)(3 - (-1))(5 - 2) = 24$, so its image under T has volume $(24)(11) = \boxed{264}$.

4. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation of rank n with standard matrix representation A. Let G be an n-box in $\mathbf{R}^n$ of volume V. Find an expression for the volume of the image of G under $T \circ T$.

   Solution. Recall (Section 2.3) that a composition of two linear transformations T and T' yields a linear transformation $T' \circ T$ having as its associated matrix the product of the matrices associated with T' and T, in that order. Thus if A is the standard matrix representation of T, then $A \cdot A = A^2$ is the standard matrix representation of $T \circ T$. Thus the volume-change factor for $T \circ T$ is $|\det(A^2)| = |(\det(A))^2| = (\det(A))^2$. Thus the volume of the image of G under $T \circ T$ is $\boxed{\det(A^2) \cdot V = (\det(A))^2 \cdot V}$.