

Sept 16.

A LOOK AT REFLECTIONS.

Suppose a line L in $\mathbb{R}^2$ contains the unit vector

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Find the matrix A of the linear transformation

$T(\vec{x}) = \text{ref}_L(\vec{x})$. Give the entries of A in terms of u_1 and u_2 . Show that A is of the form

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix}, \text{ where } a^2 + b^2 = 1.$$

Solution:

By Definition 2.2.2,

$$\text{ref}_L \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2 \left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 2(u_1 x_1 + u_2 x_2) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} (2u_1^2 - 1)x_1 + 2u_1 u_2 x_2 \\ 2u_1 u_2 x_1 + (2u_2^2 - 1)x_2 \end{bmatrix}$$

The matrix A is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2u_1^2 - 1 & 2u_1 u_2 \\ 2u_1 u_2 & 2u_2^2 - 1 \end{bmatrix}$$

Note that the sum of the diagonal entries is

$$a + d = 2(u_1^2 + u_2^2) - 2 = 0, \text{ since } \vec{u} \text{ is a unit vector. It follows that } d = -a.$$

Since $c = b$, A is of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$. Finally,

$$a^2 + b^2 = (2u_1^2 - 1)^2 + 4u_1^2 u_2^2 \\ = 4u_1^4 - 4u_1^2 + 1 + 4u_1^2(1 - u_1^2) = 1, \text{ as claimed.}$$

Suppose A is invertible. Goal: Find A^{-1} .

For example, take $n=3$.

$A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{e}_1$ has a unique solution since A is invertible.

$$\left[A \mid \vec{e}_1 \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \vec{c}_1 \\ 0 & 1 & 0 & \vec{c}_2 \\ 0 & 0 & 1 & \vec{c}_3 \end{array} \right] \left. \begin{array}{l} \text{so } A\vec{c}_1 = \vec{e}_1 \\ \text{Exact same row operations!} \\ \text{so } A\vec{c}_2 = \vec{e}_2 \\ \text{so } A\vec{c}_3 = \vec{e}_3 \end{array} \right\}$$

$$\text{So } \left[A \mid \vec{e}_1 \vec{e}_2 \vec{e}_3 \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \vec{c}_1 & \vec{c}_2 & \vec{c}_3 \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \end{array} \right],$$

$$\left[A \mid I \right] \rightarrow \left[I \mid C \right], \text{ where}$$

$$C = \left[\vec{c}_1 \vec{c}_2 \vec{c}_3 \right]$$

Question: Is C special?

Note: $AC = A[\vec{c}_1 \ \vec{c}_2 \ \vec{c}_3]$

$$= [A\vec{c}_1 \ A\vec{c}_2 \ A\vec{c}_3]$$

$$= [\vec{e}_1 \ \vec{e}_2 \ \vec{e}_3] = I$$

So $AC = I$.

But A is invertible $\rightarrow A^{-1}$ exists

$$A^{-1}AC = A^{-1}I$$

$$IC = A^{-1}I$$

$$C = A^{-1} \quad \text{Wow!}$$

Theorem: (i) An $n \times n$ matrix A is invertible iff $A \rightarrow I_n$

(ii) If A is invertible, then

$$[A \mid I] \rightarrow [I \mid A^{-1}]$$

$\nearrow$ This result gives us an easy way to find A^{-1} .

Example] (by hand) Determine if $A = \begin{bmatrix} 0 & 2 & 2 \\ 1 & -5 & 1 \\ -2 & 7 & 4 \end{bmatrix}$ is invertible. If so, find A^{-1} .

$$\left[\begin{array}{ccc|ccc} 0 & 2 & 2 & 1 & 0 & 0 \\ 1 & -5 & 1 & 0 & 1 & 0 \\ -2 & 7 & 4 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -5 & 1 & 0 & 1 & 0 \\ -2 & 7 & 4 & 0 & 0 & 1 \\ 0 & 2 & 2 & 1 & 0 & 0 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -5 & 1 & 0 & 1 & 0 \\ 0 & -3 & 6 & 0 & 2 & 1 \\ 0 & 2 & 2 & 1 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -5 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & 1 & \frac{1}{2} & 0 & 0 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -9 & 0 & -\frac{7}{3} & -\frac{5}{3} \\ 0 & 1 & -2 & 0 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 3 & \frac{1}{2} & \frac{2}{3} & \frac{1}{3} \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -9 & 0 & -\frac{7}{3} & -\frac{5}{3} \\ 0 & 1 & -2 & 0 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{6} & \frac{2}{9} & \frac{1}{9} \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 0 & \frac{1}{3} & -\frac{2}{9} & -\frac{1}{9} \\ 0 & 0 & 1 & \frac{1}{6} & \frac{2}{9} & \frac{1}{9} \end{array} \right]$$

↖ Check by hand & with Maple that this is A^{-1} .

$n=2$ is a special case (we talked about this last time)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then A^{-1} exists iff $ad-bc \neq 0$;
and when $ad-bc \neq 0$, $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Proof: Suppose $ad-bc=0$, then $ad=bc$.

If $a=c=0$, then $A = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}$ and $A\vec{x} = \vec{0}$ has infinitely many solutions, so A is not invertible.

If $a=0, c \neq 0$, then $A = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$ and columns are linearly dependent. So $A\vec{x} = \vec{0}$ has infinitely many solutions, and A is not invertible.

If $a \neq 0, c=0$, then $A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ and A is not invertible.

If $a \neq 0 \neq c$, then $ad=bc$ implies $\frac{d}{c} = \frac{b}{a} = \lambda$, a constant.

Thus, $b = \lambda a$
 $d = \lambda c \Rightarrow \begin{bmatrix} b \\ d \end{bmatrix} = \lambda \begin{bmatrix} a \\ c \end{bmatrix}$

Columns of A are dependent $\Rightarrow A\vec{x} = \vec{0}$ has infinitely many solutions. A is not invertible.

We've shown: $ad - bc = 0$ implies A is not invertible.

If $ad - bc \neq 0$, then

$$\begin{aligned} & \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

(*) You should verify that I is obtained by multiplying in reverse order.