Math 224, Fall 2007
Exam 3
Thursday, December 6, 2007

- You have 1 hour and 20 minutes.
- No notes, books, or other references.
- You are permitted to use Maple during this exam, but you must start with a blank worksheet. In particular, you are allowed to use the GramSchmidt command in Maple. Start by typing with(linalg): and with(LinearAlgebra):
- YOU MUST SHOW ALL WORK TO RECEIVE CREDIT. ANSWERS FOR WHICH NO WORK IS SHOWN WILL RECEIVE NO CREDIT (UNLESS SPECIFICALLY STATED OTHERWISE).
- Good luck! Eat candy as necessary!

Name:

"On my honor, I have neither given nor received any aid on this examination."

Signature:

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1. For parts (a)-(e), let W be the subspace of $\mathbb{R}^3$ spanned by $\{[-1,1,-1],[1,3,2]\}$.

   (a) (5 points) Find a basis for $W^\perp$.
   Solution. $W^\perp$ is the nullspace of the matrix

$$A = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 3 & 2 \end{bmatrix}.$$

   Thus

$$W^\perp = \mathrm{sp}([-5,-1,4]).$$

   (b) (5 points) Write $\mathbf{b} = [1,2,2]$ in the form

$$\mathbf{b} = \mathbf{b_W} + \mathbf{b_{W^\perp}},$$

   where $\mathbf{b_W}$ is in W and $\mathbf{b_{W^\perp}}$ is in $W^\perp$.

Solution. We want to find r_1, r_2, r_3, r_4 such that

$$r_1 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + r_2 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + r_3 \begin{bmatrix} -5 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

Thus we form the augmented matrix

$$\left[\begin{array}{ccc|c} -1 & 1 & -5 & 1 \\ 1 & 3 & -1 & 2 \\ -1 & 2 & 4 & 2 \end{array}\right]$$

and row reduce to obtain

$$r_1 = -1/3, r_2 = 11/14, r_3 = 1/42.$$

Thus

$$\mathbf{b_W} = \frac{-1}{3}[-1, 1, -1] + \frac{11}{14}[1, 3, 2] = [47/42, 85/42, 40/21]$$

and

$$\mathbf{b_{W^\perp}} = \frac{1}{42}[-5, -1, 4] = [-5/42, -1/42, 2/21].$$

(c) (5 points) Confirm that $\mathbf{b_W}$ and $\mathbf{b_{W^\perp}}$ are orthogonal.
Solution. The dot product of $\mathbf{b_W}$ and $\mathbf{b_{W^\perp}}$ is 0, so $\mathbf{b_W}$ and $\mathbf{b_{W^\perp}}$ are orthogonal.

(d) (5 points) Find the projection matrix P for W.
Solution. First construct the matrix

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 3 \\ -1 & 2 \end{bmatrix}.$$

Then the projection matrix for W is

$$P = A(A^T A)^{-1} A^T = \begin{bmatrix} 17/42 & -5/42 & 10/21 \\ -5/42 & 41/42 & 2/21 \\ 10/21 & 2/21 & 13/21 \end{bmatrix}.$$

(e) (5 points) Find $\mathbf{b_W}$ using P. Confirm that you obtain the same result for $\mathbf{b_W}$ that you did in part (b).
Solution.

$$\mathbf{b_W} = P\mathbf{b} = [47/42, 85/42, 40/21].$$

This is the same result that we obtained in part (b).

2. (16 points total, 2 points each) If the $n \times n$ matrices A and B are orthogonal, which of the following matrices must be orthogonal as well? Explain your answers.

(a) $3A$
Solution. $(3A)^T(3A) = 9A^TA = 9I \neq I$, so $3A$ is NOT orthogonal.

(b) $-B$
Solution. $(-B)^T(-B) = B^TB = I$, so $-B$ IS orthogonal.

(c) AB
Solution. $(AB)^TAB = B^TA^TAB = B^TIB = B^TB = I$, so AB IS orthogonal.

(d) $A+B$
Solution. $(A+B)^T(A+B) = (A^T+B^T)(A+B) = A^TA + A^TB + B^TA + B^TB = 2I + A^TB + B^TB \neq I$, so $(A+B)$ is NOT orthogonal.

(e) B^{-1}
Solution. Since B is orthogonal, $B^{-1} = B^T$. Thus $(B^{-1})^TB^{-1} = (B^T)^TB^T = BB^T = I$, so B^{-1} IS orthogonal.

(f) $B^{-1}AB$
Solution.

$$\begin{aligned}(B^{-1}AB)^T(B^{-1}AB) &= (B^TA^T(B^{-1})^T)(B^{-1}AB)\\ &= B^TA^T(B^T)^TB^TAB\\ &= B^TA^TBB^TAB\\ &= B^TA^TAB\\ &= I\end{aligned}$$

Thus $B^{-1}AB$ IS orthogonal.

(g) A^T
Solution. Since $A^TA = I$, $A^T = A^{-1}$, so $(A^T)^TA^T = AA^T = I$, so A^T IS orthogonal.

(h) A^2
Solution. $(A^2)^TA^2 = (AA)^TAA = A^TA^TAA = A^TA = I$, so A^2 IS orthogonal.

3. (4 points) Is the set $\{3\sin^2 x, -5\cos^2 x, 119\}$ a linearly independent set of vectors in F (the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$)?

Solution. The set is a linearly DEPENDENT set of vectors in F since, for example,

$$(\frac{119}{3})3\sin^2 x + \frac{-119}{5}(-5\sin^2 x) + (-1)119 = 0$$

is a dependence relation among the vectors.

4. (10 points) The set

$$B' = \{1+x^2, x+x^2, 1+2x+x^2\}$$

is a basis for P_2, the set of all polynomials of degree less than or equal to 2 (you do not need to show this). Find the coordinate vector of $p(x) = 1 + 4x + 7x^2$ relative to B'.

Solution. First, coordinatize the vectors in the basis B' relative to the standard basis $B = \{x^2, x, 1\}$ of P_2:

$$\begin{aligned}(1 + x^2)_B &= [1, 0, 1]\\ (x + x^2)_B &= [1, 1, 0]\\ (1 + 2x + x^2)_B &= [1, 2, 1]\end{aligned}$$

Next, note that $(p(x))_B = [7, 4, 1]$. Thus we want to find r_1, r_2, r_3 such that

$$r_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + r_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + r_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix}.$$

Thus we form the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 2 & 4 \\ 1 & 0 & 1 & 1 \end{array}\right]$$

and row reduce to obtain

$$r_1 = 2, r_2 = 6, r_3 = -1.$$

Thus the coordinate vector of $p(x)$ relative to B' is

$$(p(x))_{B'} = [2, 6, -1].$$

5. (10 points) Suppose that A is an orthogonal matrix. Show that the only eigenvalues of A are 1 and -1.

Solution. Suppose that λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$. Then

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Since A is an orthogonal matrix,

$$||A\mathbf{v}|| = ||\mathbf{v}||.$$

Thus

$$||\lambda\mathbf{v}|| = ||\mathbf{v}||.$$

But

$$||\lambda\mathbf{v}|| = |\lambda| \cdot ||\mathbf{v}||.$$

Thus
$$|\lambda| \cdot ||\mathbf{v}|| = ||\mathbf{v}||.$$
Since $\mathbf{v} \neq \mathbf{0}$ (by definition of an eigenvector), we conclude that $|\lambda| = 1$, so $\lambda = \pm 1$.

6. (a) (5 points) Let A be an $m \times n$ matrix. Show that $A^T A$ is an $n \times n$ matrix with the same rank as A. Hint: show that nullity(A) = nullity$(A^T A)$ and explain why that implies that rank(A) = rank$(A^T A)$.

Solution. Since A is an $m \times n$ matrix, A^T is an $n \times m$ matrix, so $A^T A$ is an $n \times n$ matrix. The rank equation for A is
$$\text{rank}(A) + \text{nullity}(A) = n.$$
The rank equation for $A^T A$ is
$$\text{rank}(A^T A) + \text{nullity}(A^T A) = n.$$
Thus
$$\text{rank}(A) + \text{nullity}(A) = \text{rank}(A^T A) + \text{nullity}(A^T A).$$
So to show that A and $A^T A$ have the same rank, it's sufficient to show that A and $A^T A$ have the same nullity.

- Suppose that $\mathbf{x}$ is in the nullspace of A. Then $A\mathbf{x} = \mathbf{0}$. Thus $(A^T A)\mathbf{x} = A^T A\mathbf{x} = A^T\mathbf{0} = \mathbf{0}$, so $\mathbf{x}$ is in the nullspace of $A^T A$.
- Suppose that $\mathbf{x}$ is in the nullspace of $A^T A$. Then $A^T A\mathbf{x} = \mathbf{0}$. Multiplying both sides by $\mathbf{x}^T$, we obtain:
$$\begin{aligned}
\mathbf{x}^T A^T A\mathbf{x} &= 0 \\
(A\mathbf{x})^T (A\mathbf{x}) &= 0 \\
(A\mathbf{x}) \cdot (A\mathbf{x}) &= 0 \\
||A\mathbf{x}|| &= 0 \\
A\mathbf{x} &= \mathbf{0}
\end{aligned}$$
  Thus $\mathbf{x}$ is in the nullspace of A.

We conclude that any vector in the nullspace of A is also in the nullspace of $A^T A$ and any vector in the nullspace of $A^T A$ is also in the nullspace of A. Thus the nullspace of A is equal to the nullspace of $A^T A$, so
$$\text{nullity}(A) = \text{nullity}(A^T A)$$
and
$$\text{rank}(A) = \text{rank}(A^T A).$$

(b) (5 points) Suppose that $\{\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_k}\}$ is a basis for a subspace W of $\mathbb{R}^n$. Let A be the matrix whose columns are the vectors $\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_k}$. Explain why $A^T A$ must be invertible.

Solution. Since the set $\{\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_k}\}$ is a basis for W, the vectors $\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_k}$ are independent. Thus A is an $n \times k$ matrix with rank k (A has k independent columns). Since A is an $n \times k$ matrix, A^T is a $k \times n$ matrix, so $A^T A$ is a $k \times k$ matrix. From part (a), $\text{rank}(A) = \text{rank}(A^T A)$. Thus $A^T A$ is a $k \times k$ matrix, with rank k, so it is invertible.

7. (10 points) Suppose that P is any projection matrix with columns $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$. Show that for all $i = 1, 2, \ldots, n$, $||\mathbf{v_i}||^2$ is equal to $P_{i,i}$ (the diagonal entry (i, i) in the matrix P). For example, for $i = 2$ in the matrix below, this number is $2/6 = 4/36 + 4/36 + 4/36$:

$$P = \begin{bmatrix} 5/6 & 2/6 & -1/6 \\ 2/6 & 2/6 & 2/6 \\ -1/6 & 2/6 & 5/6 \end{bmatrix}.$$

You must prove the result for an arbitrary projection matrix P; this example is just to give you an example of the result for a particular matrix.

Solution. P is a projection matrix with columns $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$:

$$P = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \cdots & \mathbf{v_n} \end{bmatrix}.$$

Then P^T is a matrix with rows $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$:

$$P^T = \begin{bmatrix} \mathbf{v_1} \\ \mathbf{v_2} \\ \cdots \\ \mathbf{v_n} \end{bmatrix}.$$

Since P is a projection matrix, $P^2 = P = P^T$. Thus

$$P^2 = P^T P = \begin{bmatrix} \mathbf{v_1} \cdot \mathbf{v_1} & \mathbf{v_1} \cdot \mathbf{v_2} & \cdots & \mathbf{v_1} \cdot \mathbf{v_n} \\ \mathbf{v_2} \cdot \mathbf{v_1} & \mathbf{v_2} \cdot \mathbf{v_2} & \cdots & \mathbf{v_2} \cdot \mathbf{v_n} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{v_n} \cdot \mathbf{v_1} & \mathbf{v_n} \cdot \mathbf{v_2} & \cdots & \mathbf{v_n} \cdot \mathbf{v_n} \end{bmatrix} = P.$$

Thus the diagonal entry

$$P_{i,i} = \mathbf{v_i} \cdot \mathbf{v_i} = ||\mathbf{v_i}||^2.$$

8. (10 points total, 1 point each) Classify each of the following statements as True or False. No explanation is necessary.

(a) The entries of an orthogonal matrix are all less than or equal to 1.
True.

(b) The determinant of any orthogonal matrix is 1.
False. The determinant can be 1 or -1.

(c) The matrix $P = A(A^T A)^{-1}A^T$ is symmetric for all matrices A.
True. For a projection matrix P, $P^T = P$.

(d) Every vector in $\mathbb{R}^n$ is in some orthonormal basis for $\mathbb{R}^n$.
False. Any vector whose norm is not equal to 1 cannot be in an orthonormal basis.

(e) Every non-zero subspace W of $\mathbb{R}^n$ has an orthonormal basis.
True. The Gram-Schmidt process enables us to construct an orthonormal basis for any subspace.

(f) Given a non-zero finite dimensional vector space V, each vector $\mathbf{v}$ in V is associated with a unique coordinate vector relative to a given basis for V.
True.

(g) Any two bases in a finite-dimensional vector space V have the same number of elements.
True.

(h) The set of all polynomials of degree 4, together with the zero polynomial, is a vector space.
False. The set is not closed under vector addition.

(i) There are only six possible ordered bases for $\mathbb{R}^3$.
False. There are infinitely many ordered bases for $\mathbb{R}^3$.

(j) There are only six possible ordered bases for $\mathbb{R}^3$ consisting of the standard unit coordinate vectors $[1, 0, 0], [0, 1, 0], [0, 0, 1]$ in $\mathbb{R}^3$.
True.

Bonus (10 points). Let P_4 denote the vector space of all polynomials of degree less than or equal to 4. We can define a dot product on P_4 by

$$p(x) \cdot q(x) = \int_{-1}^{1} p(x)q(x)\, dx.$$

Find an orthonormal basis of P_4 using this dot product.

Hint: Choose a basis B for P_4 and use the Gram-Schmidt procedure to transform your basis B into an orthonormal basis.