Linear Algebra - Exam 2 Solutions

Fall 2008 - Brad Hartlaub

#1. $f(t) = 5t^3 + 7t^2 - 8t + 9$ The first, second, and third derivatives are: $f'(t) = 15t^2 + 14t - 8$, $f''(t) = 30t + 14$, and $f'''(t) = 30$.

Suppose $\sum_{i=0}^{3} c_i f^{(i)} = \mathbf{0}$; that is,

$c_0(5t^3 + 7t^2 - 8t + 9) + c_1(15t^2 + 14t - 8) + c_2(30t + 14) + c_3(30) = 0.$

Rewritting this equation, we find

$(5c_0)t^3 + (7c_0 + 15c_1)t^2 + (-8c_0 + 14c_1 + 30c_2)t + (9c_0 - 8c_1 + 14c_2 + 30c_3) = 0$

Using the coefficients, we obtain the linear system

$$\begin{bmatrix} 5 & 0 & 0 & 0 \\ 7 & 15 & 0 & 0 \\ -8 & 14 & 30 & 0 \\ 9 & -8 & 14 & 30 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$A1$

Using Maple, we find rref $(A1) = I_4$. Thus, the set is linearly independent, $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$, and $\{f, f', f'', f'''\}$ is a basis for P_3.

#2. The coordinates c_1 and c_2 are found by writing $\vec{v}$ as a linear combination of the vectors in B. That is, we solve the equation $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

Using Maple, rref $(A2) = \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]$, so $c_1 = 3$ and $c_2 = 2$

Therefore, the coordinate vector of $\vec{v} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ relative to B is $[v]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

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#3. Let $V = \mathbb{R}^{17}$ and $W = \mathbb{R}^{32}$. If such a trasformation exists, then the rank nullity theorem implies (see p. 164)

$$\dim(V) = \dim(\operatorname{im} T) + \dim(\ker T)$$

Since $7 + 9 = 16 \neq 17$, we know that no such transformation exists.

#4. First, observe that if $p(x)$ is in P_3, then it has the form

$$p(x) = ax^3 + bx^2 + cx + d$$

so that $T(p(x)) = p'(x) = 3ax^2 + 2bx + c$.
Since $p'(x)$ is in P_2, then T is a map from P_3 to P_2.

a. To show that T is linear, let $p(x)$ and $g(x)$ be polynomials of degree 3 or less, and let k be a scalar. Recall from calc that the derivative of a sum is the sum of the derivatives, and that the derivative of a scalar times a function is the scalar times the derivative of the function. Thus,

$$\begin{aligned} T(kp(x) + g(x)) &= \frac{d}{dx}(kp(x) + g(x)) \\ &= \frac{d}{dx}(kp(x)) + \frac{d}{dx}(g(x)) \\ &= kp'(x) + g'(x) \\ &= kT(p(x)) + T(g(x)) \end{aligned}$$

Therefore, T is a linear transformation.

b. The image of the polynomial $p(x) = 3x^3 + 2x^2 - x + 2$ is

$$T(p(x)) = \frac{d}{dx}(3x^3 + 2x^2 - x + 2) = 9x^2 + 4x - 1$$

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#4 c. The only functions in P_3 with derivative equal to zero are the constant polynomials, $p(x) = c$, where c is a real number.

#5. The transformation can be written as:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ 5x + 2y \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$$

Using rref in Maple, we find

$$A^{-1} = \begin{bmatrix} 2/19 & 3/19 \\ -5/19 & 2/19 \end{bmatrix}.$$

Since A is invertible, T is an isomorphism.

Alternative Solution: The vector $\begin{bmatrix} x \\ y \end{bmatrix}$ is in the kernel of T if and only if

$$\left| \begin{matrix} 2x - 3y = 0 \\ 5x + 2y = 0 \end{matrix} \right|$$

This linear system has the unique solution $x = y = 0$. Thus $\ker(T) = \{\vec{0}\}$ and T is an isomorphism.

#6. a. By denoting the vectors in $\mathfrak{B}$ by $\vec{v}_1$ and $\vec{v}_2$ and those in $\mathfrak{B}^*$ by $\vec{v}_1^*$ and $\vec{v}_2^*$, the column vectors of the transition matrix are $[\vec{v}_1]_{\mathfrak{B}^*}$ and $[\vec{v}_2]_{\mathfrak{B}^*}$. These coordinate vectors are found from the equations

$$c_1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } d_1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + d_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Using Maple (rref(A_6) and rref(C_6)) to solve these equations, we find $c_1 = 2$, $c_2 = 3$ and $d_1 = 0$, $d_2 = -1$, so $[\vec{v}_1]_{\mathfrak{B}^*} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $[\vec{v}_2]_{\mathfrak{B}^*} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. The transition matrix is $S = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}$

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#6 b. Since $[\vec{v}]_{B^*} = S[\vec{v}]_B$,

$$[\vec{v}]_{B^*} = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \end{bmatrix}$$

c. Yes, the coordinates are different, but $\vec{v}$ is the same no matter which basis you choose. For example,

$$3\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} = 6\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 11\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

7. To find the image of $\vec{v}$, we first write the vector as a linear combination of the basis vectors. In this case,

$$\vec{V} = \vec{e}_1 + 3\vec{e}_2 + 2\vec{e}_3.$$ Applying T to this linear combination and using the linear properties of T, we find

$$\begin{aligned} T(\vec{v}) &= T(\vec{e}_1 + 3\vec{e}_2 + 2\vec{e}_3) \\ &= T(\vec{e}_1) + 3T(\vec{e}_2) + 2T(\vec{e}_3) \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 9 \end{bmatrix} \end{aligned}$$

#8 a. Using our projection formula, we find

$$\vec{u}_1 = \frac{1}{\|\vec{v}\|}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{Proj}_{\vec{v}}(\vec{u}) = (\vec{u}_1 \cdot u)\,\vec{u}_1 = \left(\frac{1}{\sqrt{2}}(1) + \frac{1}{\sqrt{2}}(3)\right)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \left(\frac{4}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

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#8. b. $\vec{u} - \text{Proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

To show that $\text{proj}_{\vec{v}}(\vec{u})$ is orthogonal to $\vec{u} - \text{proj}_{\vec{v}}(\vec{u})$ we compute the dot product.

$$\text{Proj}_{\vec{v}}(\vec{u}) \cdot (\vec{u} - \text{proj}_{\vec{v}}(\vec{u})) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (2)(-1) + 2(1) = 0$$

Since the dot product is zero, the vectors are orthogonal.

#9. Let $\vec{w} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$. Thus, any vector in W has the form $c\vec{w}$, for some real c. Consequently, a vector $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ is orthogonal to W if and only if $\vec{v} \cdot \vec{w} = 0$ or

$x - 2y + 3z = 0$. The solution set to this equation is $S = \{(2s - 3t, s, t) : s, t \in \mathbb{R}\}$

Therefore, the set of vectors orthogonal to W is given by

$$S = \left\{ \begin{bmatrix} 2s - 3t \\ s \\ t \end{bmatrix}, s, t \in \mathbb{R} \right\} = \left\{ s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, s, t \in \mathbb{R} \right\}$$

Letting $s = t = 1$ gives the particular vector

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},$$

which is orthogonal to W since

$$\vec{v} \cdot \vec{w} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = (-1)(1) + 1(-2) + (1)(3)$$
$$= 0$$

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#10. The first normalized vector is $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

The next vector in nonnormalized form is

$$\vec{v}_2^{\perp} = v_2 - v_2^{\parallel} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1$$

$$= \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} - \frac{2}{\sqrt{10}}\frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 18/10 \\ 2 \\ -6/10 \end{bmatrix}$$

The second normalized vector is

$$\vec{u}_2 = \frac{1}{\|\vec{v}_2^{\perp}\|}\vec{v}_2^{\perp} = \frac{1}{\sqrt{\left(\frac{18}{10}\right)^2 + \frac{400}{100} + \frac{36}{100}}}\begin{bmatrix} 18/10 \\ 20/10 \\ -6/10 \end{bmatrix}$$

$$= \frac{1}{\sqrt{\frac{760}{100}}}\begin{bmatrix} 18/10 \\ 20/10 \\ -6/10 \end{bmatrix} = \frac{\sqrt{190}}{38}\begin{bmatrix} 18 \\ 20 \\ -6 \end{bmatrix}$$

The third vector in nonnormalized form is

$$\vec{v}_3^{\perp} = \vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2$$

$$= \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} - \frac{9}{\sqrt{10}}\frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \frac{\sqrt{190}}{38}(54+20-12)\begin{bmatrix} 18 \\ 20 \\ -6 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 9/10 \\ 0 \\ 27/10 \end{bmatrix} - \frac{31\sqrt{190}}{19}\begin{bmatrix} 18 \\ 20 \\ -6 \end{bmatrix}$$

Finally

$$\vec{u}_3 = \frac{1}{\|\vec{v}_3^{\perp}\|}\vec{v}_3^{\perp} \underset{\text{use maple}}{=} \begin{bmatrix} 3/\sqrt{19} \\ -3/\sqrt{19} \\ -1/\sqrt{19} \end{bmatrix}$$

OR simply use GramSchmidt([a1, a2, a3], normalized=true);

to obain

$$\begin{bmatrix} \sqrt{10}/10 \\ 0 \\ \frac{3\sqrt{10}}{10} \end{bmatrix}, \begin{bmatrix} \frac{9}{190}\sqrt{190} \\ \frac{1}{19}\sqrt{190} \\ -\frac{3}{190}\sqrt{190} \end{bmatrix}, \begin{bmatrix} \frac{3}{19}\sqrt{19} \\ -\frac{3}{19}\sqrt{19} \\ -\frac{1}{19}\sqrt{19} \end{bmatrix}$$

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#11 The system is inconsistent, so we use the normal equations: $A^T A \vec{x} = A^T \vec{b}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

Carrying out the multiplication leads to

$$\begin{bmatrix} 14 & 32 \\ 32 & 77 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$$

Thus, premultiplying both sides by $(A^T A)^{-1}$ we find

$$\vec{x} = \begin{bmatrix} -5/2 \\ 1 \end{bmatrix}$$

#12. The determinant of the coefficient matrix is given by

$$\det \begin{bmatrix} 2 & 3 \\ -5 & 7 \end{bmatrix} = 14 - (-15) = 29$$

Since the det $\neq 0$, the system has a unique solution which is

$$x = \frac{\det \begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix}}{29} = \frac{14-9}{29} = \frac{5}{29}$$

and

$$y = \frac{\det \begin{bmatrix} 2 & 2 \\ -5 & 3 \end{bmatrix}}{29} = \frac{6-(-10)}{29} = \frac{16}{29}.$$

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#13 a. $d_n = \det \begin{bmatrix} 0 & \\ 0 & M_{n-1} \\ \vdots & \\ 1 & \end{bmatrix}$

Now, if we expand down the first column, we find that $d_n = \det(M_{n-1})$ if n is odd or $d_n = -\det(M_{n-1})$ if n is even. Thus, $d_n = (-1)^{(n-1)} d_{n-1}$.

b. $d_1 = 1$, $d_2 = -1$, $d_3 = -1$, $d_4 = 1$, $d_5 = 1$, $d_6 = -1$, $d_7 = -1$, $d_8 = 1$
(switching between -1 and 1 with every other increase)

c. The periodicity in part b suggests $d_{n+4} = d_n$
Thus, $d_{100} = d_4 = 1$.

#14. No, the determinant of the sum is not the sum of the determinants.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Then $\det(A) = 1 = \det(B)$

However, $\det(A+B) = \det\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4$

$\neq 1$.