Math 224, Fall 2007
Exam 1

You have 1 hour and 20 minutes.

No notes, books, or other references.

You are permitted to use the Maple worksheet MapleCommands.mw located on the P: drive.

YOU MUST SHOW ALL WORK TO RECEIVE CREDIT

Good luck!

Name: Solutions.

"On my honor, I have neither given nor received any aid on this examination."

Signature:

Question	Score	Maximum
1		12
2		12
3		10
4		16
5		8
6		10
7		12
8		10
9		10
Bonus		10
Total		100

1. Solve the following system of equations in the variables x, y, z, w:

$$\begin{aligned} x - y + z + w &= 5 \\ y - z + 2w &= 8 \\ 2x - y - 3z + 4w &= 18 \end{aligned}$$

Solution. First we form the augmented matrix $[A|\mathbf{b}] = \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 5 \\ 0 & 1 & -1 & 2 & 8 \\ 2 & -1 & 3 & 4 & 18 \end{array}\right]$. Row reducing, we obtain $rref([A|\mathbf{b}]) = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 13 \\ 0 & 1 & 0 & 2 & 8 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$. The fourth column does not contain a pivot, so w is a free variable. We set $w = r$. Then we obtain:

$$\begin{aligned} x &= 13 - 3r \\ y &= 8 - 2r \\ z &= 0 \\ w &= r \end{aligned}$$

Thus the solution of the system of equations is given by:

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = r \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 13 \\ 8 \\ 0 \\ 0 \end{bmatrix}.$$

2. Find a basis for (a) the nullspace, (b) the column space, and (c) the row space of the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 4 \\ 1 & 2 & 1 & 1 & 6 \\ 0 & 1 & 1 & 1 & 3 \\ 2 & 2 & 0 & 1 & 7 \end{bmatrix}$$

Solution. $rref(A) = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$

(a) To find a basis for the nullspace of A, we must solve $A\mathbf{x} = \mathbf{0}$. Since the third and fifth columns of $rref(A)$ do not contain pivots, x_3 and x_5 are free variables. We set $x_3 = r$ and $x_5 = s$. Then we obtain:

$$\begin{aligned} x_1 1 &= r - s \\ x_2 &= -r - 2s \\ x_3 &= r \\ x_4 &= -r \\ x_5 &= s \end{aligned}$$

Thus a basis for the nullspace of A is the set of vectors

$$\{\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}\}.$$

(b) A basis for the column space of A consists of the columns of A corresponding to the columns of $rref(A)$ with pivots:

$$\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\}.$$

(c) A basis for the row space of A consists of the non-zero rows of $rref(A)$:

$$\{[1, 0, -1, 0, 1], [0, 1, 1, 0, 2], [0, 0, 0, 1, 1]\}.$$

3. Is the set $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} \text{ such that } x_1 + x_2 + \ldots + x_n = 0 \right\}$ a subspace of $\mathbf{R}^n$? Note: be sure to justify your response.

Solution. Let S denote the set $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} \text{ such that } x_1 + x_2 + \ldots + x_n = 0 \right\}$.

To prove that S is a subspace of $\mathbf{R}^n$, we must verify that S is closed under vector addition and scalar multiplication. So, let $\mathbf{u} = [u_1, u_2, \ldots, u_n]$ and $\mathbf{v} = [v_1, v_2, \ldots, v_n]$ be two vectors in S, and let r be any scalar. Now, $\mathbf{u} + \mathbf{v} = [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n]$. Since $\mathbf{u}$ and $\mathbf{v}$ are in S, $u_1 + u_2 + \ldots + u_n = 0$ and $v_1 + v_2 + \ldots + v_n = 0$. Thus $u_1 + v_1 + u_2 + v_2 + \ldots + u_n + v_n = (u_1 + u_2 + \ldots + u_n) + (v_1 + v_2 + \ldots + v_n) = 0 + 0 = 0$, so $\mathbf{u} + \mathbf{v}$ is in S. Similarly, $r\mathbf{u} = [ru_1, ru_2, \ldots, ru_n]$, and $ru_1 + ru_2 + \ldots + ru_n = r(u_1 + u_2 + \ldots + u_n) = r \cdot 0 = 0$, so $r\mathbf{u}$ is in S. Thus S is a subspace of $\mathbf{R}^n$.

4. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be a linear transformation such that $T([1,0,0]) = [1,2,1]$, $T([0,1,0]) = [3,0,4]$, and $T([1,0,1]) = [5,4,6]$.

   Solution.

   (a) Find the standard matrix representation of T.
   $T([0,0,1]) = T([1,0,1]) - T([1,0,0]) = [5,4,6] - [1,2,1] = [4,2,5]$. Thus the standard matrix representation of T is

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 2 \\ 1 & 4 & 5 \end{bmatrix}.$$

   (b) Use the standard matrix representation to find a formula for $T([x_1, x_2, x_3])$.

$$T([x_1, x_2, x_3]) = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 + 4x_3 \\ 2x_1 + 2x_3 \\ x_1 + 4x_2 + 5x_3 \end{bmatrix}$$

   (c) Find the kernel of T.
   To find the kernel of T, we solve the system $A\mathbf{x} = \mathbf{0}$.

$$rref(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

   Since the third column does not contain a pivot, x_3 is a free variable, and we set $x_3 = r$. Then $x_1 = -r$, $x_2 = -r$, and $x_3 = r$, so

$$ker(T) = sp(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}).$$

   (d) Is the linear transformation T invertible? If so, find the standard matrix representation of T^{-1}.
   T is not invertible since A is not row equivalent to I_3.

5. Suppose that T is a linear transformation with standard matrix representation A, and that A is a 7×6 matrix such that the nullspace of A has dimension 4. What is the dimension of the range of T?

   Solution. Since the nullity of A is equal to 4, the rank of A is equal to 2. Thus the dimension of the range of T is 2.

6. Is the following set of vectors dependent or independent?

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Solution. The matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & -5 & 0 \\ -2 & 3 & 1 \end{bmatrix}$ is row equivalent to I_3 (i.e. $rref(A) = I_3$), so the vectors are independent.

7. If a 7×9 matrix A has rank 5, find the dimension of the column space of A, the dimension of the nullspace of A, and the dimension of the row space of A.

   Solution. The dimension of the column space of A is 5, the dimension of the nullspace of A is 4, and the dimension of the row space of A is 5.

8. Suppose that the vectors $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{x}$ are mutually perpendicular (i.e. $\mathbf{v}$ and $\mathbf{w}$ are perpendicular, $\mathbf{v}$ and $\mathbf{x}$ are perpendicular, and $\mathbf{w}$ and $\mathbf{x}$ are perpendicular). Use dot products to find $||\mathbf{v}+3\mathbf{w}+2\mathbf{x}||$ in terms of the magnitudes (lengths) of $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{x}$. Hint: Start by computing $||\mathbf{v}+3\mathbf{w}+2\mathbf{x}||^2$.

Solution.

$$\begin{aligned}||\mathbf{v}+3\mathbf{w}+2\mathbf{x}||^2 &= (\mathbf{v}+3\mathbf{w}+2\mathbf{x})\cdot(\mathbf{v}+3\mathbf{w}+2\mathbf{x})\\ &= \mathbf{v}\cdot\mathbf{v}+\mathbf{v}\cdot 3\mathbf{w}+\mathbf{v}\cdot 2\mathbf{x}+3\mathbf{w}\cdot\mathbf{v}+3\mathbf{w}\cdot 3\mathbf{w}+3\mathbf{w}\cdot 2\mathbf{x}+2\mathbf{x}\cdot\mathbf{v}+2\mathbf{x}\cdot 3\mathbf{w}+2\mathbf{x}2\mathbf{x}\\ &= ||\mathbf{v}||^2+9||\mathbf{w}||^2+4||\mathbf{x}||^2\end{aligned}$$

Thus

$$||\mathbf{v}+3\mathbf{w}+2\mathbf{x}|| = \sqrt{||\mathbf{v}||^2+9||\mathbf{w}||^2+4||\mathbf{x}||^2}.$$

9. Classify each of the following statements as True or False. No explanation is necessary.

Solution.

(a) If A is a 2×3 matrix and B is a 2×4 matrix, then AB is a 3×4 matrix.
False. AB is undefined.

(b) Any six vectors in $\mathbf{R}^4$ must span $\mathbf{R}^4$.
False. The statement is only true if 4 of the vectors are linearly independent.

(c) Every independent subset of $\mathbf{R}^n$ is a subset of some basis for $\mathbf{R}^n$.
True. Any independent subset of $\mathbf{R}^n$ can be enlarged to form a basis for $\mathbf{R}^n$.

(d) If A is a 7×4 matrix, and if the dimension of the column space of A is 3, then the columns of A are linearly dependent.
True. Since the rank of A is not equal to the number of columns of A, the columns of A are linearly dependent.

(e) If T is a linear transformation, then $T(\mathbf{0}) = \mathbf{0}$.
True.

Bonus Question. Suppose we have three matrices A, B_1, and B_2, with the following properties:

1. A is a 4×4 matrix, B_1 is a 4×3 matrix, and B_2 is a 3×4 matrix.
2. $A\mathbf{v} = B_1(B_2\mathbf{v})$ for all vectors $\mathbf{v}$ in $\mathbf{R}^4$

Show that there must exist a non-zero vector in the nullspace of A, and that there must also exist a vector in $\mathbf{R}^4$ which is not in the column space of A.

Solution. If $B_2\mathbf{v} = \mathbf{0}$, then $A\mathbf{v} = \mathbf{0}$, so the nullspace of A is contained in the nullspace of B_2. Since B_2 is a 3×4 matrix, there can be at most 3 pivots, so there is at least one free variable. Thus the nullspace of B_2 must contain a non-zero vector, so the nullspace of A must as well. Next, any linear combination of columns of A is also a linear combination of columns of B_1, so the column space of A is contained in the columns space of B_1. Since B_1 is a 4×3 matrix, there can be at most 3 pivots, so the dimension of the row space of B_1 is at most 3, so there must be a row of zeros in $rref(B_1)$. Thus, there must be some $\mathbf{b} \in \mathbf{R}^4$ such that $B_1\mathbf{x} = \mathbf{b}$ does not have a solution. Thus $\mathbf{b}$ is not in the column space of B_1, so $\mathbf{b}$ is not in the column space of A since the column space of A is contained in the column space of B_1.