Linear Algebra Exam 1 Solutions
Brad Hartlaub - Fall 2008.

Part 1 #1.

$$\left|\begin{array}{l} x+y+z=4 \\ -x-y+z=-2 \\ 2x-y+2z=2 \end{array}\right| \xrightarrow[-2E_1+E_3 \to E_3]{E_1+E_2 \to E_2} \left|\begin{array}{l} x+y+z=4 \\ \quad 2z=2 \\ -3y \quad =-6 \end{array}\right|$$

$$\xrightarrow[\text{interchange } E_2 \text{ and } E_3]{} \left|\begin{array}{l} x+y+z=4 \\ -3y \quad =-6 \\ \quad 2z=2 \end{array}\right|$$

Using back substitution, we have $z=1$, $y=2$, and $x=4-y-z$ or $x=4-2-1=1$. Therefore, the system is consistent with the unique solution $(1, 2, 1)$.

#2.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 2 & 1 & 5 & b \\ 1 & -1 & 1 & c \end{array}\right] \xrightarrow[-R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & 1 & b-2a \\ 0 & -1 & -1 & c-a \end{array}\right]$$

$$\xrightarrow{R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & 1 & b-2a \\ 0 & 0 & 0 & b+c-3a \end{array}\right]$$

Hence, the corresponding linear system is consistent provided that $b+c-3a=0$. That is, the system is consistent for all a, b, and c such that the point (a, b, c) lies on the plane $b+c-3a=0$. Notice also that when the system is consistent, the third row will contain all zeros and the variable x_3 is a free variable.

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Part 1 #3 Recall: Def 2.1.1 A function (or mapping) T from $\mathbb{R}^2$ to $\mathbb{R}^3$ is called a linear transformation (or map) if there is a 3×2 matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$

$$f(s,t) = f\left(\begin{bmatrix} s \\ t \end{bmatrix}\right) = \begin{bmatrix} 2s+3t \\ -s+5t \\ 4s-3t \end{bmatrix} = s\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} + t\begin{bmatrix} 3 \\ 5 \\ -3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 2 & 3 \\ -1 & 5 \\ 4 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} s \\ t \end{bmatrix}}_{\vec{x}}$$

Therefore, $f(s,t)$ is a linear map.

#4. Start by letting B denote an arbitrary 2×2 matrix

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then the product of matrix A on the left with matrix B on the right is given by $AB = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix}$

On the other hand, $BA = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a+b & b \\ c+d & d \end{bmatrix}$

Setting $AB = BA$, we obtain

$$a = a+b, \quad a+c = c+d, \quad \text{and} \quad b+d = d.$$

So that $b=0$ and $a=d$. Let S be the set of all 2×2 matrices defined by

$$S = \left\{ \begin{bmatrix} a & 0 \\ c & a \end{bmatrix} , a, c \in \mathbb{R} \right\}.$$ Then each

matrix in S commutes with A

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Part 1 #5.

$$A = \begin{bmatrix} 1 & 3 & 3 & -1 & 2 & 17 \\ 2 & 6 & -2 & 14 & -3 & -19 \\ 4 & 12 & 2 & 16 & 1 & 7 \end{bmatrix} \xrightarrow[-4R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \begin{bmatrix} 1 & 3 & 3 & -1 & 2 & 17 \\ 0 & 0 & -8 & 16 & -7 & -53 \\ 0 & 0 & -10 & 20 & -7 & -61 \end{bmatrix}$$

$$\xrightarrow[R_3-R_2 \to R_3]{} \begin{bmatrix} 1 & 3 & 3 & -1 & 2 & 17 \\ 0 & 0 & -8 & 16 & -7 & -53 \\ 0 & 0 & -2 & 4 & 0 & -8 \end{bmatrix} \xrightarrow[-R_3 \div 2 \to R_3]{R_2 \times (-1) \to R_2} \begin{bmatrix} 1 & 3 & 3 & -1 & 2 & 17 \\ 0 & 0 & +8 & -16 & +7 & 53 \\ 0 & 0 & 1 & -2 & 0 & 4 \end{bmatrix}$$

$$\xrightarrow[8R_3-R_2 \to R_3]{} \begin{bmatrix} 1 & 3 & 3 & -1 & 2 & 17 \\ 0 & 0 & 8 & -16 & 7 & 53 \\ 0 & 0 & 0 & 0 & -7 & -21 \end{bmatrix} \xrightarrow[-R_3 \to R_3]{} \begin{bmatrix} \boxed{1} & 3 & 3 & -1 & 2 & 17 \\ 0 & 0 & \boxed{8} & -16 & 7 & 53 \\ 0 & 0 & 0 & 0 & \boxed{7} & 21 \end{bmatrix}$$

This row echelon form shows that we have 3 nonzero pivots. Thus, the rank of A is 3. Many students will continue until they get reduced row echelon form, but that is not necessary.

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Part 2 #1. The general form of a parabola is given by $y = ax^2 + bx + c$. Conditions on a, b, and c are imposed by substituting the given points into this equation. This gives

$$1 = a(-1)^2 + b(-1) + c = a - b + c$$
$$-2 = a(2)^2 + b(2) + c = 4a + 2b + c$$
$$1 = a(3)^2 + b(3) + c = 9a + 3b + c$$

From these conditions, we obtain the linear system

$$\left\{ \begin{array}{l} a - b + c = 1 \\ 4a + 2b + c = -2 \\ 9a + 3b + c = 1 \end{array} \right|$$

Now, we use row operations to eliminate a from equations 2 and 3.

$$\left| \begin{array}{l} a - b + c = 1 \\ 4a + 2b + c = -2 \\ 9a + 3b + c = 1 \end{array} \right| \xrightarrow[-9E_1 + E_3 \to E_3]{-4E_1 + E_2 \to E_2} \left| \begin{array}{l} a - b + c = 1 \\ \quad 6b - 3c = -6 \\ \quad 12b - 8c = -8 \end{array} \right|$$

Now, eliminate b from equation 3.

$$\xrightarrow{-2E_2 + E_3 \to E_3} \left| \begin{array}{r} a - b + c = 1 \\ 6b - 3c = -6 \\ -2c = 4 \end{array} \right|$$

Using back substitution, we find $c = -2$, $b = -2$, and $a = 1$. Thus, the parabola of interest is

$$\boxed{y = x^2 - 2x - 2}$$

Completing the square gives the parabola in standard form

$$y = (x-1)^2 - 3,$$ so the vertex is $(1, -3)$

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Part 2 #2. The vector $\vec{v}$ is a linear combination of the vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ if there are scalars c_1, c_2, and c_3 such that

$$\begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}$$

The augmented matrix corresponding to this equation is given by

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{array}\right]$$

Row Reduction yields

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

Thus, the linear system is inconsistent. Therefore, the vector $\vec{v}$ cannot be written as a linear combination of the three vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

#3. a. $$AB = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & -1 \\ 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 16 & 3 \\ 3 & 6 & 2 \end{bmatrix}$$

b. $$AB = \underbrace{\begin{bmatrix} 1 & 3 & 2 & -5 \\ 2 & 2 & -3 & 4 \\ 5 & 1 & 1 & 6 \end{bmatrix}}_{3\times 4} \underbrace{\begin{bmatrix} 2 & 3 \\ -5 & 0 \\ 4 & 1 \end{bmatrix}}_{3\times 2} = \text{Not defined.}$$

(In short, we say that these matrices are not compatible.)

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Part 2 #4. Consider the augmented matrix

$$\left[\begin{array}{ccc|c} -1 & 4 & -6 & a \\ 2 & 1 & 3 & b \\ 1 & -3 & 5 & c \end{array}\right],$$ which describes a linear system.

Reducing the augmented matrix, we find

$$\left[\begin{array}{ccc|c} -1 & 4 & -6 & a \\ 0 & 9 & -9 & b+2a \\ 0 & 0 & 0 & c + \frac{7}{9}a - \frac{1}{9}b \end{array}\right]$$

Thus, the original system is consistent only if

$7a - b + 9c = 0$. This is the equation of a plane in 3-space, and hence the span is not all of $\mathbb{R}^3$. Notice that the solution to the equation $7a - b + 9c = 0$ can be written in parametric form by letting $b = r$ and $c = s$

so $a = \frac{1}{7}r - \frac{9}{7}s$, so that

$$\operatorname{Span}\left\{\begin{bmatrix}-1\\2\\1\end{bmatrix}, \begin{bmatrix}4\\1\\-3\end{bmatrix}, \begin{bmatrix}-6\\3\\5\end{bmatrix}\right\} = \left\{ r\begin{bmatrix}1/7\\1\\0\end{bmatrix} + s\begin{bmatrix}-9/7\\0\\1\end{bmatrix}, \ s,t \in \mathbb{R} \right\}$$

Thus, the span is the subspace of all linear combinations of two linearly independent vectors, highlighting the geometric interpretation of the solution as a plane

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Part 2 #5. We seek solutions to the vector equation

$$c_1\begin{bmatrix}1\\0\\2\end{bmatrix}+c_2\begin{bmatrix}-1\\1\\2\end{bmatrix}+c_3\begin{bmatrix}-2\\3\\1\end{bmatrix}+c_4\begin{bmatrix}2\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

This leads to the linear system

$$\left|\begin{array}{l} c_1 - c_2 - 2c_3 + 2c_4 = 0 \\ \quad c_2 + 3c_3 + c_4 = 0 \\ 2c_1 + 2c_2 + c_3 + c_4 = 0 \end{array}\right|$$

Row-reducing the coefficient matrix

$$A5 = \begin{bmatrix}1 & -1 & -2 & 2\\0 & 1 & 3 & 1\\2 & 2 & 1 & 1\end{bmatrix}$$

gives

$$\operatorname{rref}(A5) = \begin{bmatrix}1 & 0 & 0 & 2\\0 & 1 & 0 & -2\\0 & 0 & 1 & 1\end{bmatrix}$$

Thus, the vectors are linearly dependent and

$$\vec{v}_4 = 2\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3.$$

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Part 2 #6. a. Since $rref(A_6) = I_3$, the $Ker(A) = 0$

b. A basis for the image of A is $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

c. The augmented matrix

$$C6 = \left[\begin{array}{ccc|c} 1 & -1 & -2 & 3 \\ -1 & 2 & 3 & 1 \\ 2 & -2 & -2 & -2 \end{array}\right]$$ row reduces to

$$rref(C_6) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -4 \end{array}\right].$$

Thus, the linear system $A\vec{x} = \vec{b}$ is consistent with solution

$$\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 8\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} - 4\begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$$

i.e., $\vec{x} = \begin{bmatrix} 3 \\ 8 \\ -4 \end{bmatrix}$.

d. Yes, the matrix is invertible.

$$rref(D6) = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1/2 \\ 0 & 1 & 0 & 2 & 1 & -1/2 \\ 0 & 0 & 1 & -1 & 0 & 1/2 \end{array}\right]$$

Thus, $$A^{-1} = \begin{bmatrix} 1 & 1 & 1/2 \\ 2 & 1 & -1/2 \\ -1 & 0 & 1/2 \end{bmatrix}$$

e. $Rank(A) = 3$, since $rref(A6) = I_3$.

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Part2 #7

$$A = \begin{array}{c} \begin{array}{cccccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 & \vec{v}_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The columns with leading 1's form a basis for the image of A. Thus, a basis of the image of A is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

To form a basis for the kernel of A, notice that

$$2\vec{v}_2 - \vec{v}_3 = 0, \quad \vec{v}_1 = 0$$

and

$$3\vec{v}_2 + 4\vec{v}_4 + 5\vec{v}_5 - \vec{v}_6 = 0.$$

Using Kyle numbers for these relations, a basis for the kernel of A is

$$\left\{ \begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 4 \\ 5 \\ -1 \end{bmatrix} \right\}.$$