Math 347

How Busy Must a Road Be to Require a Pedestrian Crossing Control?

Suppose that the arrival of cars that pass a point on a road where pedestrians cross is a Poisson process with mean λ cars per second. Assume that the cars all travel in the same direction, as in each lane of a two-lane road with a median.

1. Let G_j denote the time gap (often called the interarrival time) between the $(j-1)$-th car and the j-th car. Find

$$\mathbb{P}(G_j > t).$$

   Note (primarily for the next question) that since the arrival of cars follows a Poisson process, the interarrival times G_j are *independent*.

2. Let T be the time that it takes a typical pedestrian to cross the road. Show that the probability that a typical pedestrian crosses the road during the j-th time gap, i.e. between the $(j-1)$-th car and the j-th car, is

$$(1 - e^{-\lambda T})^{j-1} e^{-\lambda T}.$$

   Hint: Express this probability in terms of $G_0, G_1, \ldots, G_{j-2}, G_{j-1}, G_j$ and use independence.

3. Show that the average number of gaps that a typical pedestrian requires to cross the road (including the gap during which he crosses) is $e^{\lambda T}$. Hint: You'll need to use differentiation of the geometric series.

4. Let α denote the average time length of the gaps that occur while a typical pedestrian is waiting to cross. Explain why the typical pedestrian's average waiting time is

$$(e^{\lambda T} - 1)\alpha.$$

5. Now, let τ be the longest time for which, on average, a typical pedestrian can reasonably be expected to wait. Then a pedestrian crossing control is needed UNLESS

$$(e^{\lambda T} - 1)\alpha \le \tau.$$

6. Next, note that whatever the value of α (the average time length of gaps that occur) is, it cannot exceed T, for then at least one time gap would have exceeded T, and the pedestrian would have been able to cross. Thus a sufficient condition for the previous inequality is

$$(e^{\lambda T} - 1)T \le \tau.$$

7. Solve the previous inequality for λ in terms of τ and T. This expresses the maximum rate at which cars should be allowed to pass if pedestrians are not to be delayed unreasonable, or, because long delays encourage pedestrians to take risks, as the maximum safe traffic flow above which a pedestrian crossing control is required.

8. It is convenient to convert this quantity into number of cars per hour and express it as a function of the width of the road. Suppose that τ is 60 seconds. Led d be the width of the road in feet, and assume that a pedestrian walks at a rate of 3.5 feet per second. Find the maximum safe traffic flow ϕ in cars per hour as a function of road width d. Graph ϕ as a function of d. Points in the plane below $\phi(d)$ represent safe traffic flows, and points above represent unsafe ones in which a pedestrian crossing control is required. Discuss and interpret the results. What is the maximum safe traffic flow if the road is 20 feet wide? 30 feet wide? How sensitive are your results to the maximum reasonable waiting time τ? Experiment with different values of τ and discuss your results.

When Does a T-junction Require a Left-turn Lane?

Consider a T-junction on a narrow two-lane road. Suppose that the major road runs from north to south, with the minor road leaving to the west. There is no left-turn lane on the major road at the junction. Thus, if southbound traffic is heavy, cars turning left may have difficulty entering the minor road and may cause delays to northbound traffic. When is this problem serious enough to warrant building a left-turn lane?

Assume that the arrival of cars traveling from north to south is a Poisson process with mean λ cars per second, and that the arrival of cars traveling from south to north is a Poisson process with mean μ cars per second. Let α denote the probability that one of the northward bound cars will turn left at the junction; hence $1-\alpha$ is the probability that it will continue forward. Let T be the minimum time gap in the southbound traffic flow that will allow a northbound car to turn left (assumed the same for all cars). We will assume that drivers joining the main road at the junction from the west interrupt neither the northbound nor the southbound flow, but rather enter the main stream of traffic only when it is completely safe to do so. We will also assume that drivers act responsibly, so that southbound cars are not impeded by drivers turning left when such a move would be dangerous.

1. Let the random variable S denote the time gap in seconds between southbound cars. Show that
$$\mathbb{P}(S > t) = e^{-\lambda t},$$
$0 < t < \infty$.

2. Let the random variable N denote the time gap in seconds between northbound cars when northbound traffic is moving freely. Show that
$$\mathbb{P}(N > t) = e^{-\mu t},$$
$0 < t < \infty$.

3. Suppose that a northbound car has reached the T-junction and is waiting to turn left. Let the random variables $S_1, S_2, S_3, \ldots$ denote the sequence of time gaps in the southbound flow which confronts this driver, and let U_j be the event that she turns left (i.e. enters the minor road) during the j-th such gap. Since the gaps are independent, her situation is identical to that of a crossing pedestrian. Use this argument to find $\mathbb{P}(U_j)$.

4. Let V_j be the random variable defined by
$$V_j = S_1 + S_2 + \cdots + S_j.$$
Show that the probability density function of V_j is
$$f_{V_j}(v) = \frac{\lambda e^{-\lambda v}(\lambda v)^{j-1}}{(j-1)!}.$$

5. Show that
$$\mathbb{P}(V_j) < N = \left(\frac{\lambda}{\lambda+\mu}\right)^j.$$

6. Define a *tailback* to be group of one or more northbound cars that have been brought to a half behind a car turning left at the junction. Let W denote the event that NO TAILBACK forms behind the driver waiting to turn left. Given that our driver enters the minor road during gap j (i.e. event U_j occurs), the probability that no tailback forms behind her is simply the probability that the $(j-1)$-th southbound car passes her before the next northbound car passes her at the junction. Use this argument to show that
$$\mathbb{P}(W|U_j) = \left(\frac{\lambda}{\lambda+\mu}\right)^{j-1}.$$

7. Show that
$$\begin{aligned}\mathbb{P}(W) &= \sum_{j=1}^{\infty} \mathbb{P}(W|U_j) \cdot \mathbb{P}(U_j) \\ &= e^{-\lambda T} \sum_{j=1}^{\infty} \left(\frac{\lambda(1-e^{-\lambda T})}{\lambda+\mu}\right)^{j-1}\end{aligned}$$

8. Let θ denote the probability that a tailback DOES form behind a left-turning driver. Show that
$$\theta = \frac{\mu(1-e^{-\lambda T})}{\mu + \lambda e^{-\lambda T}}.$$
Note that because the formation of a tailback is governed by two memoryless Poisson processes, θ is also the probability that a tailback containing two or more cars will form behind a tailback containing precisely one car.

9. We shall now consider four traffic configurations, labeled $i = 0, 1, 2, 3$.

Configuration	Last car	Status of northbound lane
$i = 0$	Has continued forward	No tailback
$i = 1$	Is waiting to turn left	No tailback
$i = 2$	Has continued forward	Tailback
$i = 3$	Is waiting to turn left	Tailback

Find the transition probabilities p_{ij}, where p_{ij} is the probability that j is the next configuration if the current one is i. Use the transition probabilities to form the transition matrix P.

10. Find the stationary distribution of the T-junction.

11. Find the stationary proportion of configurations in which a tailback exists in the northbound lane. This should be a function of λ, μ, T, α. Denote this function by $\Gamma(\lambda, \mu, T, \alpha)$. Experiment with different values of λ, μ, T, α to obtain different numerical results for Γ. Remember what represent when choosing your numbers. Two examples that you can work with are the following. The junction of High Road (the major road, which runs north and south) with Hartsfield Road (the minor road, which runs east and west) in Tallahassee, Florida (Mesterton-Gibbons 1987b, *Mathematical and Computer Modelling*, **9**, 625-629) is a T-junction like the one we have considered here. On April 10, 1986, the City of Tallahassee Traffic Engineering Department enumerated peak-hour traffic movements for one hour during the morning rush hour and one hour during the evening rush hour. In the morning hour, 437 vehicles traversed the junction in the southbound direction and 154 in the northbound direction, of which 41 turned left. Use these measurements to obtain values of λ, μ, α for the morning rush hour. In the evening hour, 383 vehicles traversed the junction in the southbound direction and 337 in the northbound direction, of which 133 turned left. Use these measurements to obtain values of λ, μ, α for the evening rush hour. Find the stationary proportion of configurations in which a tailback exists in the northbound lane for each of these situations. No data for T were available, though it seems reasonable to assume, for example, $T = 6$. To determine whether a left-turn lane is necessary, a group of traffic engineers and/or city officials would presumably determine a maximum proportion of traffic configurations in which a tailback is deemed acceptable.