Generating Functions

Generating functions possess a beautiful and rich theory as well as powerful utility. Entire books are written and graduate courses are offered on the topic, so we'll only scratch the surface this evening in our introduction to generating functions. If you want to study further, I recommend you look at the book *Generatingfunctionology* by Herbert S. Wilf. One of the most important uses of generating functions is in solving recursion problems, and that will be our focus tonight. Once you get the hang of this method, I guarantee you'll see how fun and addicting these problems can be. HAVE FUN!

Definition Suppose (a_n) is a sequence of numbers for $n = 0, 1, 2, ...$ The *ordinary generating function* $A(x)$ for the sequence (a_n) is the power series

$$A(x) \;=\; \sum_{n=0}^{\infty} a_n x^n \;.$$

The *exponential generating function* for (a_n) is the series

$$B(x) \;=\; \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \;.$$

Skeleton of the Method

1. Recurrence relation:

$$a_{n+1} \;=\; 3a_n + 5\;, \quad n \geq 0, \quad a_0 = 1\;.$$

2. Convert to series:

$$\sum_{n=0}^{\infty} a_{n+1} x^n \;=\; \sum_{n=0}^{\infty} 3a_n x^n + \sum_{n=0}^{\infty} 5x^n, \quad a_0 = 1\;.$$

3. Express in terms of generating function $A(x)$:

$$\frac{1}{x}\left[A(x) - 1\right] \;=\; 3A(x) + \frac{5}{1-x}\;.$$

4. Isolate $A(x)$ via algebra:

$$A(x) \;=\; \frac{1+4x}{(1-x)(1-3x)}\;.$$

5. Invert $A(x)$ back into the world of power series:

$$A(x) \;=\; \sum_{n=0}^{\infty} \left(\frac{7}{2}3^n - \frac{5}{2}\right) x^n\;.$$

6. Extract from $A(x)$ the coefficients a_n:

$$a_n \;=\; \frac{7}{2}3^n - \frac{5}{2}\;.$$

Cast of Common Characters

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{n=0}^{\infty} n x^{n-1}$$

$$\frac{2}{(1-x)^3} = \frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) = \sum_{n=0}^{\infty} n(n-1) x^{n-2}$$

$$\frac{1}{1-\theta x} = \sum_{n=0}^{\infty} \theta^n x^n \quad , \ \theta \neq 0$$

$$\frac{\theta}{(1-\theta x)^2} = \frac{d}{dx}\left(\frac{1}{1-\theta x}\right) = \sum_{n=0}^{\infty} n \theta^n x^{n-1} \quad , \ \theta \neq 0$$

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\exp(\theta x) = 1 + \theta x + \frac{\theta^2 x^2}{2!} + \frac{\theta^3 x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{\theta^n x^n}{n!}$$

$$\left(\sum_{k=0}^{\infty} a_k x^k\right) \cdot \left(\sum_{j=0}^{\infty} b_j x^j\right) = \sum_{n=0}^{\infty} c_n x^n \quad \text{where} \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Good Practice Problems on Generating Functions

1. Find a closed-form formula for the terms in the sequence (a_n) if this sequence is defined by the following recurrence relation

$$a_{n+1} = 2a_n + 4^n , \quad n = 0, 1, 2, ..., \quad a_0 = 1.$$

2. Find a formula in terms of n of the general term r_n for the sequence defined by the recurrence relation

$$r_{n+1} = r_n + n + 1 , \quad n = 0, 1, 2, ..., \quad r_0 = 1 .$$

3. Derive a formula in terms of n for the general term w_n for the sequence defined by the recurrence relation

$$w_n = 3w_{n-1} + 2w_{n-2} , \quad n \geq 2 ,$$

with initial values $w_0 = 1$, and $w_1 = 3$.

4. Consider the sequence (t_n) determined by the recursive definition

$$t_{n+1} = 3t_n - t_{n-1} , \quad n = 1, 2, 3, ..., \quad t_0 = 0, \ t_1 = 1 .$$

(a) Find an explicit formula for t_n in terms of n for $n = 0, 1, 2, ...$

(b) Show that for large n, $t_n \approx C \cdot b^n$, where C and b are real constants, and determine the values of C and b.

5. Consider the following recurrence relation

$$a_{n+1} = na_n + 2 , \quad n = 0, 1, 2, ..., \quad a_0 = 1.$$

(a) Find a closed-form formula for a_n.

(b) Show that for large n

$$a_n \approx \alpha(n-1)! ,$$

and find the value of the constant α.

6. A *derangement* of n objects is a permutation of the objects that leaves no object in its original position. The number of possible derangements on a set of n items follows the recurrence

$$d_{n+1} = n(d_{n-1} + d_n) \ , \ n \geq 1 , \quad d_0 = 1, \ d_1 = 0 .$$

(a) Find a formula for d_n in terms of n.

(b) If n objects are permuted at random, then the probability of a derangement is given by $d_n/n!$. Evaluate

$$\lim_{n\to\infty} \frac{d_n}{n!} ,$$

whose value is the asymptotic probability that when n objects are permuted, that none of them remain in their original position.

7. Government A sends secret messages to its ally Government B in the form of codewords. Suppose a codeword of length n is a sequence of n digits with each digit being an element of the set $\{0, 1, 2, 3, 4, 5\}$. As a small measure of security, consider a codeword legitimate if and only if there are an even number of 0's in the word (consider a codeword containing no 0's as legitimate.) Therefore, 0000, 1050, and 5423 are all legitimate codewords of length four, while 4035 is not legitimate. Illegitimate words are ignored by the intended receiver as noise, but their presence will confuse spies and eavesdroppers. Let a_n denote the number of possible legitimate codewords of length n that can be formed using this code scheme.

(a) If we set $a_0 = 1$ by convention, show that

$$a_{n+1} = 4a_n + 6^n , \quad n = 0, 1, 2, ..., \quad a_0 = 1$$

is a recurrence relation for a_n.

(b) Derive a formula for a_n as a function of n.

8. Let $b_n = 1^2 + 2^2 + 3^2 + \cdots + n^2$ denote the sum of the first n squares. The sequence (b_n) naturally satisfies the recurrence relation

$$b_{n+1} = b_n + (n+1)^2 , \quad n = 0, 1, 2, ..., \quad b_0 = 0.$$

(a) Show that the ordinary generating function for the sequence (b_n) is

$$B(x) = \frac{x(1+x)}{(1-x)^4} .$$

(b) By inverting the generating function $B(x)$, find a closed-form formula for b_n.

9. Consider a binary random number generator which ouputs a string of n 0's and 1's, where each digit in the string is equally likely to be a 0 or 1, independent of other digits in the string. For example, if this random string of digits were to be observed after twenty digits generated, we might observe the following random binary sequence,

00001101101111100011 .

Find the probability that after n digits have been generated, the resulting binary string has no two 1's in a row?