# Math 347

## 1.4 #4 Solution

(a) We will use the following variables in the model:

| | |
|---|---|
| $n$ | number of additional crews to be brought in |
| $t$ | time (in days) to clean up the spill |
| $T$ | total cost of the clean up (in dollars) |
| $F$ | amount of the fine (in dollars) |

We are assuming that each crew (i.e. the local crew and any additional crews that are brought in) clean at the same rate. Since there are 200 square miles of beach to clean and each crew can clean 5 square miles per week, we have

$$200 = \frac{5}{7}(n+1)t.$$

This gives us an algebraic relationship between the number of additional crews to be brought in ($n$) and the total time required for the clean up ($t$). The fine $F$ is a function of time $t$:

$$F(t) = \begin{cases} 0 & \text{if } t \leq 14 \\ 10000(t-14) & \text{if } t > 14 \end{cases}.$$

The total cost of the clean up is given by

$$T(t) = 500t + (18000 + 800t)n + F.$$

Since we are trying to determine the number of additional crews to bring in to minimize the total cost of the clean up, it makes sense to write $T$ as a function of $n$. We use the equation $200 = \frac{5}{7}(n+1)t$ to obtain $t = \frac{280}{n+1}$. Substituting this expression for $t$ in our equation for $T$, we obtain

$$T(n) = 500\frac{280}{n+1} + (18000 + 800\frac{280}{n+1})n + F.$$

When $t = 14$, $n = 19$, so we can rewrite our model in the following way:

$$\begin{aligned} T_1(n) &= 500\tfrac{280}{n+1} + (18000 + 800\tfrac{280}{n+1})n, \quad n \geq 19 \\ T_2(n) &= 500\tfrac{280}{n+1} + (18000 + 800\tfrac{280}{n+1})n + 10000(\tfrac{280}{n+1} - 14), \quad n < 19 \end{aligned}$$

Note that $t \leq 14$ corresponds to $n \geq 19$ since $t$ decreases as $n$ increases, and $t > 14$ corresponds to $n < 19$. Since we essentially have two different models, we need to determine which will minimize total cost. We can do this by minimizing $T_1(n) =$

$500\frac{280}{n+1}+(18000+800\frac{280}{n+1})$ over all $n \geq 19$ and minimizing $T_2(n) = 500\frac{280}{n+1}+(18000+800\frac{280}{n+1})n + 10000(\frac{280}{n+1} - 14)$ over all $n < 19$ and comparing the resulting costs to determine the global minimizer.

As seen in the accompanying Maple worksheet (see oil.mw), the minimum of $T_1$ occurs at $n = 19$, and the corresponding minimal cost is $T_1(19) = \$561800$. The minimum of $T_2$ occurs at $n = 11.28$. The corresponding minimal costs for 11 and 12 additional crews are $T_2(11) = \$508333$ and $T_2(12) = \$508923$. We see that $T_2(11)$ is the overall minimum. Thus the company should bring in 11 additional crews to clean up the oil spill. The cost to the company will be approximately \$508333. Note that this means that more than 14 days will be required to clean up the spill (since the minimum cost occurs when we use the $T_2$ model), However, it is important to note that this is not an assumption; rather, we deduce this information from the analysis of our models.

(b) Next, we consider the sensitivity of our model with respect to the rate at which a crew can clean up the shoreline. We generalize the model by replacing the rate 5/7 with a parameter $\alpha$:

$$200 = \alpha(n + 1)t.$$

For values of $\alpha$ near $\frac{5}{7}$, we expect that the minimum should still occur in the interval $n \leq 19$, so we use the model $T_2$ to compute the sensitivity. See the Maple worksheet oil.mw for the computations of the sensitivity calculations. We obtain $S(n^\star, \alpha) = -0.54$, so that if the cleanup crews are 10% faster than expected, the optimal number of crews decreases by about 5.4%.

(c) Next, we consider the sensitivity of our model with respect to the amount of the fine. We generalize our equation for the fine to be $F = \beta(t - 14)$ if $t > 14$ and $F = 0$ if $t \leq 14$. For values of $F$ near 10,000, we use the model $T_2(n) = 500\frac{280}{n+1} + (18000 + 800\frac{280}{n+1})n + \beta 10000(\frac{280}{n+1} - 14)$. See the Maple worksheet oil.mw for the computations of the sensitivity calculations. We first compute the optimal number of crews in terms of $\beta$, and then use the equation $200 = \frac{5}{7}(n + 1)t$ to obtain an expression for the optimal time in terms of $\beta$. We can then compute $\frac{dt^\star}{d\beta}$ and find the sensitivity $S(t^\star, \beta) = \frac{dt^\star}{d\beta}\frac{\beta}{t^\star}$. We obtain $S(t^\star, \beta) = -0.52$, so that if the fine is raised by 2%, then the cleanup time should decrease by about 1%. Substituting the optimal value $n^\star$ into the equation for the total cost $T$, we can compute the sensitivity of the total cost of the cleanup to the fine. We find $S(T^\star, beta) = 0.17$, so that if the fine is increased then the total cost of the cleanup will increase by about 1.7% for every \$1,000 per day increase in the fine.

(d) This question is really open to interpretation. On the one hand, the fine is only 19% of the total cost, and if the fine were reduced by 50%, then the number of days to clean up the spill would increase by about 25%, and it would only save the company about 8.5% of the total cost. So in some sense, the amount of the fine does not seem excessive. On the other hand, if the 14 day limit were extended to 21 days, cleanup would proceed exactly as before, only the company would save $70,000. So in some sense, the 14 day limit does seem excessive.