Math 333
Higher Order Linear Differential Equations
Nonhomogeneous Equations: The Method of Undetermined Coefficients

Recall that the general solution of the nonhomogeneous n-th order linear differential equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t)$$

is given by the sum

$$y_h(t) + y_p(t),$$

where $y_h(t)$ is the general solution of the associated homogeneous equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0$$

and $y_p(t)$ is one particular solution of the nonhomogeneous equation. Using the method of the characteristic polynomial, we have seen how to find $y_h(t)$ for the case in which the differential equation has constant coefficients:

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = 0.$$

We can obtain a particular solution $y_p(t)$ of the nonhomogeneous n-th order linear differential equation using the method of undetermined coefficients, provided that $g(t)$ is of an appropriate form.

Examples.

1. Find the general solution of the differential equation

$$y''' - 3y'' + 3y' - y = 4e^t.$$

Solution. The general solution of the associated homogeneous equation is

$$y_h(t) = c_1e^t + c_2te^t + c_3t^2e^t.$$

To find a particular solution $y_p(t)$, we start by assuming that $y_p(t)$ is of the form $y_p(t) = \alpha e^t$. However, since e^t, te^t, and t^2e^t are all solutions of the homogeneous equation, we must multiply our initial choice by t^3. Thus, our final assumption is that $y_p(t)$ is of the form

$$y_p(t) = \alpha t^3e^t.$$

Substituting y_p and its derivatives into the differential equation and equating terms, we obtain $\alpha = 2/3$. Thus, the general solution of the differential equation is

$$y(t) = c_1e^t + c_2te^t + c_3t^2e^t + \frac{2}{3}t^3e^t.$$

2. Find the general solution of the differential equation

$$y^{(4)} + 2y'' + y = 3\sin t - 5\cos t.$$

Solution. The general solution of the associated homogeneous equation is

$$y_h(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t.$$

Our initial assumption for a particular solution is $y_p(t) = \alpha \sin t + \beta \cos t$, but we must multiply this initial choice by t^2. Thus, our final assumption is that $y_p(t)$ is of the form

$$y_p(t) = \alpha t^2 \sin t + \beta t^2 \cos t.$$

Substituting y_p and its derivatives into the differential equation and equating terms, we obtain $\alpha = -3/8$ and $\beta = 5/8$. Thus, the general solution of the differential equation is

$$y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t + \frac{-3}{8} t^2 \sin t + \frac{5}{8} t^2 \cos t.$$

3. Find the general solution of the differential equation

$$y''' - 4y' = t + 3\cos t + e^{-2t}.$$

Solution. The general solution of the associated homogeneous equation is

$$y_h(t) = c_1 + c_2 e^{2t} + c_3 e^{-2t}.$$

Recall that we can write a particular solution of the differential equation as a sum of particular solutions of the differential equations

$$\begin{aligned} y''' - 4y' &= t, \\ y''' - 4y' &= 3\cos t, \\ y''' - 4y' &= e^{-2t}. \end{aligned}$$

Our initial choice for a particular solution of the first equation is $y_1(t) = At + B$, but since a constant is a solution of the homogeneous equation, we multiply by t. Thus, we choose

$$y_1(t) = At^2 + Bt.$$

For the second equation, we choose

$$y_2(t) = C\cos t + D\sin t.$$

For the third equation, we choose

$$y_3(t) = Ete^{-2t}.$$

We obtain $A = -1/8$, $B = 0$, $C = 0$, $D = -3/5$, and $E = 1/8$. Thus, the general solution of the differential equation is

$$y(t) = c_1 + c_2 e^{2t} + c_3 e^{-2t} - \frac{1}{8} t^2 - \frac{3}{5} \sin t + \frac{1}{8} te^{-2t}.$$