Math 333
Higher Order Linear Differential Equations

The theoretical structure and methods of solution that we developed for second order linear differential equations extend directly to linear differential equations of third and higher order.

Definition. An n-th order linear differential equation is a differential equation of the form

$$P_0(t)\frac{d^n y}{dt^n} + P_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + P_{n-1}(t)\frac{dy}{dt} + P_n(t)y = G(t). \tag{1}$$

We assume that the functions $P_0,\ P_1, \ldots, P_n$, and $G(t)$ are continuous real-valued function on some interval I and that P_0 is nowhere zero in this interval. Then, dividing Eqn. (1) by $P_0(t)$, we obtain

$$\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t). \tag{2}$$

Since the differential equation in Eqn. (2) involves the n-th derivative of y with respect to t, it will require n integrations to solve Eqn. (2). Each of these integrations introduces an arbitrary constant, so the general solution of Eqn. (2) will contain n arbitrary constants. Thus, to obtain a unique solution, it is necessary to specify n initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \ldots, y^{(n-1)}(t_0) = y_0^{(n-1)}, \tag{3}$$

where t_0 may be any point in the interval I.

Theorem: Existence and Uniqueness of Solutions. If the functions $p_1,\ p_2, \ldots, p_n$, and g are continuous on the open interval I, then there exists exactly one solution $y = \varphi(t)$ that satisfies the differential equation (2) and the initial conditions (3).

The Homogeneous Equation. As in the case of second order linear differential equations, we'll first consider the homogeneous equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0. \tag{4}$$

If the functions $y_1,\ y_2, \ldots, y_n$ are solutions of Eqn. (4), then it follows by direct computation that the **linear combination**

$$y(t) = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t)$$

is also a solution of Eqn. (4). In fact, every solution of Eqn. (4) can be expressed in this form, and the general solution of Eqn. (4) is

$$y(t) = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t),$$

where $y_1,\ y_2, \ldots, y_n$ is a **linearly independent** set of solutions of Eqn. (4).

Example. Verify that the functions $1,\ t,\ t^3$ are solutions of the third order linear differential equation $ty^{(3)} - y'' = 0$. Find the general solution of the differential equation.

Note: Read this subsection if you have had a course in linear algebra, or if you are interested in the general theory. See me if you have questions or if you would like more details.

Definition. The functions $f_1,\ f_2, \ldots, f_n$ are said to be **linearly dependent** on an interval I if there exists a set of constants $k_1, k_2, \ldots, k_n$, not all zero, such that

$$k_1f_1 + k_2f_2 + \cdots + k_nf_n = 0$$

for all t in I. If no such constants exist, then the functions $f_1,\ f_2, \ldots, f_n$ are said to be **linearly independent**.

It can be shown that a necessary and sufficient condition for the solutions $y_1,\ y_2, \ldots, y_n$ to be linearly independent is that $W(y_1, y_2, \ldots, y_n) \neq 0$, where $W(y_1, y_2, \ldots, y_n)$ is the Wronskian defined as

$$W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdots & \cdots & & \cdots \\ \cdots & \cdots & & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

The Nonhomogeneous Equation. Next consider the nonhomogeneous differential equation given by Eqn. (2),

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t).$$

Suppose that $y_h(t)$ is the general solution of the homogeneous equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0,$$

and suppose that $y_p(t)$ is one particular solution of the nonhomogeneous equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t).$$

Then the general solution of the nonhomogeneous equation is

$$y(t) = y_h(t) + y_p(t).$$

The Homogeneous Equation with Constant Coefficients. Consider the n-th order linear homogeneous differential equation with constant coefficients:

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0. \tag{5}$$

Based on our knowledge of second order linear differential equations with constant coefficients, it is natural to anticipate that $y = e^{rt}$ will be a solution of Eqn. (5) for suitable values of r. Substituting $y = e^{rt}$ into the differential equation, we obtain

$$e^{rt}(a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n) = 0.$$

The polynomial

$$Z(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n \tag{6}$$

is called the **characteristic polynomial** or **characteristic equation** of the differential equation. This polynomial of degree n has n zeros, say $r_1,\ r_2, \ldots, r_n$. Note that some of the r_i may be equal, and that some may be complex.

- **Real and Unequal Roots.** If the roots of the characteristic equation are real and no two are equal, then the general solution of Eqn. (5) is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{rt}. \tag{7}$$

  Example. Find the general solution of the differential equation

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0.$$

  Solution. The roots of the characteristic polynomial are $r_1 = 1,\ r_2 = -1,\ r_3 = 2,\ r_4 = -3$. Thus, the general solution of the differential equation is

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-3t}.$$

- **Complex Roots.** If the characteristic polynomial has complex roots, they must occur in conjugate pairs $\lambda \pm \mu i$. Provided that none of the roots is repeated, the general solution of Eqn. (5) is still of the form given in Eqn. (7). However, just as for the second order equation, we can replace the complex-valued solutions $e^{\lambda + i\mu t}$ and $e^{\lambda - i\mu t}$ by the real-valued solutions

$$e^{\lambda t}\cos(\mu t) \text{ and } e^{\lambda t}\sin(\mu t).$$

Example. Find the general solution of the differential equation

$$y^{(4)} - y = 0.$$

Solution. The roots of the characteristic polynomial are $r = 1, \ -1, \ i, \ -i$. Thus, the general solution of the differential equation is

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t.$$

- **Repeated Roots.** If the roots of the characteristic polynomial are not distinct, i.e. if some of the roots are repeated, then the solution of the form given in Eqn. (7) is not the general solution of Eqn. (5). Recall that if r_1 is a repeated root of the second order linear equation $ay'' + by' + cy = 0$, then two solutions are $e^{r_1 t}$ and $te^{r_2 t}$. For a differential equation of order n, if a root of $Z(r) = 0$, say $r = r_1$ has multiplicity s, then

$$e^{r_1 t}, \ \ te^{r_1 t}, \ \ t^2 e^{r_1 t}, \ldots, t^{s-1} e^{r_1 t}$$

are corresponding solutions of Eqn. (5). If a complex root $\lambda + i\mu$ is repeated s times, then its complex conjugate $\lambda - i\mu$ must also be repeated s times. Corresponding to these $2s$ complex-valued roots, we can find $2s$ real-valued solutions

$$e^{\lambda t}\cos(\mu t), \ e^{\lambda t}\sin(\mu t), \ te^{\lambda t}\cos(\mu t), \ te^{\lambda t}\sin(\mu t), \ldots, t^{s-1}e^{\lambda t}\cos(\mu t), \ t^{s-1}e^{\lambda t}\sin(\mu t).$$

Example. Find the general solution of the differential equation

$$y^{(4)} + 2y'' + y = 0.$$

Solution. The roots of the characteristic polynomial are $r = i, \ i, \ -i, \ -i$. Thus, the general solution of the differential equation is

$$y(t) = c_1 \cos(t) + c_2 \sin(t) + c_3 t \cos(t) + c_4 t \sin(t).$$

Example. Find the general solution of the differential equation

$$y^{(6)} - 3y^{(4)} + 3y'' - y = 0.$$

Solution. The roots of the characteristic polynomial are $r = -1, \ -1, \ -1, 1, 1, 1$. Thus, the general solution of the differential equation is

$$y(t) = c_1 e^{-t} + c_2 te^{-t} + c_3 t^2 e^{-t} + c_4 e^t + c_5 te^t + c_6 t^2 e^t.$$