Math 347
Thursday, November 8, 2007
Chaos and the Lorenz Equations

Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?

These problems are due on Thursday, November 15, 2007.

1. Consider a layer of air that is heated from the bottom. The warmer air rising up interacts with the colder air sinking down, and rolling convection is formed. Let x represent the rate at which the convection rolls rotate. You can think of x as measuring the speed of motion in the air due to convection. Let y represent the temperature difference between the ascending and descending air currents, and let z represent the deviation from the linearity of the vertical temperature profile (a positive value of z indicates that temperature varies faster near the boundary). You can think of z as a measure of the vertical temperature difference as you move through the system from top to bottom. The dynamical system that follows is derived from the physical laws that govern convection, and was studied by mathematician and meteorologist E. Lorenz.

$$\begin{aligned}\frac{dx}{dt} &= -\sigma x + \sigma y\\ \frac{dy}{dt} &= -y + rx - xz\\ \frac{dz}{dt} &= -bz + xy\end{aligned}$$

The constants σ, r, and b are physical parameters of the system. We will consider the classic case in which $\sigma = 10$ and $b = 8/3$. The parameter r represents the temperature difference between the top and bottom of the air layer. Increasing r pumps more energy into the system, creating more vigorous dynamics. We'll start by considering the classic case $r = 28$, but later you should experiment with different values of r.

(a) Find the equilibrium values of the system. Discuss the stability of the equilibrium values. Confirm graphically that the equilibrium values that you find are actually equilibrium values.

(b) Graphically simulate $z(t)$ vs. $x(t)$ for the initial conditions $(2, 5, 20)$ over the time range $0 \le t \le 10$.

(c) Graphically simulate $z(t)$ vs. $x(t)$ for the initial conditions $(2.1, 5, 20)$ over the time range $0 \le t \le 10$.

(d) Numerically simulate the system for the two different sets of initial conditions described above.

(e) Plot $x(t)$ vs. t on the same set of axes for the two different sets of initial conditions described above. Do the same for $y(t)$ vs. t and $z(t)$ vs. t.

(f) Now, you should notice that something strange and exciting seems to be going on here. We'd really like to be able to watch the trajectories for $x(t)$, $y(t)$, and $z(t)$ as they are being drawn for different initial conditions. Unfortunately, Maple does not seem to have the ability to animate DEplots. Use the Lorenz Attractor Java applet from Washington and Lee University (accessible through the daily objectives page) to simulate the system. Using this applet, you can watch simultaneous trajectories being drawn for different initial conditions. Click on *Launch Lorenz Attractor* to start the applet. Once the applet starts, you can change the initial conditions and parameters to simulate the system. Leave the Δt box unchanged. You can change the amount of time that the simulations run for by increasing n. Leave the Simultaneous Trajectories box checked. The Deviation field lets you change the difference in the initial conditions. So Deviation=0.1 corresponds to the difference in initial conditions that we plotted in Maple.

(g) Discuss the results. What happens? Is chaos totally random? Or is there a deterministic component? Is there any structure to the results that you observe? One of the properties used to describe chaos is *sensitive dependence on initial conditions*. Based on your observations, discuss the meaning of this property. What is the meaning of the statement *Does the flap of a butterfly's wings in Brazil set off a tornado in Texas*?

(h) Simulate the system for other initial conditions and other values of r, using both Maple and the Java applet, and see if you find any interesting results.

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2. ([MM] 6.5 #20). This problem illustrates the striking difference between the behavior of continuous-time and discrete-time dynamical systems that can occur even in simple models.

(a) Show that the continuous-time dynamical system

$$\begin{aligned}\frac{dx}{dt} &= (\alpha-1)x - \alpha x^2\\ \frac{dy}{dt} &= x - y\end{aligned}$$

has a stable equilibrium at $x = y = \frac{\alpha-1}{\alpha}$ for any $\alpha > 1$. Graphically and numerically simulate the system for $\alpha = 1.5,\ 2.0,\ 2.5,\ 3.0,\ 3.5, 4.0$.

(b) Consider the analogous discrete-time dynamical system

$$\begin{aligned}x(t+1) - x(t) &= (\alpha-1)x(t) - \alpha(x(t))^2\\ y(t+1) - y(t) &= x(t) - y(t)\end{aligned}$$

Show that this system also has an equilibrium at $x = y = \frac{\alpha-1}{\alpha}$ for any $\alpha > 1$. Use graphical and numerical simulation to explore the stability of the equilibrium and the behavior of nearby solutions. For each of the cases $\alpha = 1.5,\ 2.0,\ 2.5,\ 3.0,\ 3.5, 4.0$, try several different initial conditions near the equilibrium point and report what you see.

(c) Discuss the differences between the continuous and discrete systems.