Putnam Problem-Solving Seminar
l'Hôpital's Rule

l'Hôpital's rule can be used to compute limits containing indeterminate forms. Suppose $f(x)$ and $g(x)$ are differentiable and that $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \text{ and } \lim_{x\to a} g(x) = \pm\infty.$$

In other words, we have an indeterminate form of type $0/0$ or $\pm\infty/\pm\infty$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

1. Evaluate
$$\lim_{x\to\infty} \left(\frac{1}{x}\frac{a^x-1}{a-1}\right)^{1/x},$$
where $a > 1$.

2. Evaluate
$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n.$$

3. Evaluate
$$\lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^n.$$

4. Evaluate
$$\lim_{n\to\infty} \left(1+\frac{1}{n^2}\right)^n.$$

5. Evaluate
$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n^2}.$$

6. Evaluate
$$\lim_{n\to\infty} \frac{2p_nP_n}{p_n+P_n},$$
where
$$p_n = \left(1+\frac{1}{n}\right)^n$$
and
$$P_n = \left(1+\frac{1}{n}\right)^{n+1}.$$

7. Evaluate
$$\lim_{n\to\infty} 4^n \left(1 - \cos\frac{\theta}{2^n}\right).$$

8. Suppose that f is a function with two continuous derivatives and $f(0) = 0$. Prove that the function g defined by $g(0) = f'(0)$, $g(x) = f(x)/x$ for $x \neq 0$, has a continuous derivative.

9. Let $0 < a < b$. Evaluate
$$\lim_{t\to 0}\left[\int_0^1 [bx + a(1-x)^t]\,dt\right]^{1/t}.$$

10. Calculate
$$\lim_{x\to\infty} x\int_0^x e^{t^2-x^2}\,dt.$$

11. Prove that the function $y = (x^2)^x$, $y(0) = 1$, is continuous at $x = 0$.

12. Evaluate
$$\lim_{x\to 0}\frac{1}{x}\int_0^x (1+\sin(2t))^{1/t}\,dt.$$

Some other fun problems related to Calculus.

1. Let $f : [0, 1] \to (0, 1)$ be continuous. Show that the equation
$$2x - \int_0^x f(t)\,dt = 1$$
has one and only one solution in the interval $[0, 1]$.

2. Suppose that f is a continuous function for all x which satisfies the equation
$$\int_0^x f(t)\,dt = \int_1^x t^2 f(t)\,dt + \frac{x^{16}}{8} + \frac{x^{18}}{9} + C,$$
where C is a constant. Find an explicit form for $f(x)$ and find the value of the constant C.

3. Suppose that f is differentiable and that $f'(x)$ is strictly increasing for $x \geq 0$. If $f(0) = 0$, prove that $\frac{f(x)}{x}$ is strictly increasing for $x > 0$.

4. Find all the differentiable functions f defined for $x > 0$ which satisfy $f(xy) = f(x) + f(y)$, $x, y > 0$.