General Theory: Series Solutions Near an Ordinary Point

We have considered the problem of finding solutions of the differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{1}$$

where P, Q, and R are polynomials, in the neighborhood of an ordinary point x_0. Recall that an ordinary point is any point x_0 such that $P(x_0) \neq 0$.

Theorem. If x_0 is an ordinary point of Eqn. (1), then there exists a solution $y(x)$ of Eqn. (1) of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 y_1(x) + a_1 y_2(x),$$

where a_0 and a_1 are arbitrary and y_1 and y_2 are linearly independent solutions that have Taylor series expansions around x_0. Moreover, the radius of convergence of the series solution $y(x)$ is *at least as large as* the minimum of the radii of convergence of the power series for $p = Q/P$ and $q = R/P$.

Example 1. Consider the differential equation

$$y'' + x^2 y' + (1 + x^2)y = 0.$$

Can we determine a series solution about $x = 0$ for the differential equation? If so, what is the radius of convergence? For this differential equation, $p(x) = x^2$ and $q(x) = 1 + x^2$. Both $p(x)$ and $q(x)$ have Taylor series expansions about $x_0 = 0$ that converge for all x. Thus, there is a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

of the differential equation that converges for all x.

Example 2. Consider the differential equation

$$(1 - x^2)y'' - 2xy' + 2y = 0.$$

Can we determine a series solution about $x = 0$ for the differential equation? If so, what is the radius of convergence? For this differential equation, $p(x) = \dfrac{-2x}{1 - x^2}$ and $q(x) = \dfrac{2}{1 - x^2}$. Both $p(x)$ and $q(x)$ have Taylor series expansions about $x_0 = 0$ that converge for $-1 < x < 1$. Thus, there is a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

of the differential equation that converges at least for $-1 < x < 1$, and possibly for larger values of x. (In fact, it can be shown that the series solution converges for all x).