Taylor's Theorem: Accuracy for Taylor Polynomials

Recall that the n-th order Taylor polynomial for a function $f(x)$ centered at x_0 is given by:

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \\ &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \end{aligned}$$

Previously, we observed that the Taylor polynomial $P_n(x)$ approximates $f(x)$ around x_0, and that the accuracy of the approximation improves as n increases. Taylor's theorem provides a precise statement of how close P_n is to the actual function f:

Taylor's Theorem. Suppose that $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$ is the n-th order Taylor polynomial for $f(x)$ centered at x_0. Suppose that for all x in some interval I,

$$|f^{(n+1)}(x)| \leq K_{n+1}.$$

Then

$$|f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!}|x-x_0|^{n+1}.$$

This theorem says that the maximum possible error committed by P_n in approximating f is given by the expression $\frac{K_{n+1}}{(n+1)!}|x-x_0|^{n+1}$.

Examples.

1. Suppose that the 4-th order Maclaurin polynomial is used to approximate the function $f(x) = e^x$. Find the maximum error committed by the polynomial in approximating e^2. What order polynomial must be used in order to approximate e^2 to within 10^{-10}?
2. Find the Taylor polynomials P_1 and P_2 for the function $f(x) = \sqrt{x}$ centered at $x_0 = 64$. Find the maximum errors committed by these polynomials in approximating $\sqrt{65}$ and $\sqrt{80}$.
3. Find the Maclaurin polynomial P_5 for the function $f(x) = \sin(x)$. Suppose that we want to use this polynomial as an approximation for $f(x) = \sin x$ for all x in the interval $[-2, 2]$. Find the maximum error committed by P_5 in approximating f on the interval $[-2, 2]$.