Practice Problems: Taylor and Maclaurin Series

Solutions

1. $f(x) = e^x$ centered at $x = 3$.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(3)}{k!}(x-3)^k$$

$f(3) = e^3$ $f'(3) = e^3$, etc. For all k, $f^{(k)}(3) = e^3$

$$\boxed{\sum_{k=0}^{\infty} \frac{e^3}{k!}(x-3)^k}$$

2. $f(x) = e^{5x}$, centered at $x = 0$.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$$

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	e^{5x}	$e^0 = 1$
1	$5e^{5x}$	$5e^0 = 5$
2	$5^2 e^{5x}$	5^2
3	$5^3 e^{5x}$	5^3

In general,

$f^{(k)}(0) = 5^k$

$$\Rightarrow \boxed{\sum_{k=0}^{\infty} \frac{5^k}{k!}x^k}$$

3. $f(x) = \sin x$ centered at $x = \pi/2$.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(\pi/2)}{k!} \left(x - \frac{\pi}{2}\right)^k$$

k	$f^{(k)}(x)$	$f^{(k)}(\pi/2)$
0	$\sin x$	1
1	$\cos x$	0
2	$-\sin x$	-1
3	$-\cos x$	0
4	$\sin x$	1

For even k, $f^{(k)}(\pi/2) = \pm 1$

For odd k, $f^{(k)}(\pi/2) = 0$

This cycle repeats.

$$\boxed{1 - \frac{1}{2!}(x - \pi/2)^2 + \frac{1}{4!}(x - \pi/2)^4 - \frac{1}{6!}(x - \pi/2)^6 + \cdots = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{(2k)!}(x - \pi/2)^{2k}}$$

4. $\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$

$$\Rightarrow f(x) = \cos(\pi x) = \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{(\pi x)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} (-1)^k \frac{\pi^{2k} x^{2k}}{(2k)!}}$$

5. $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

$$\Rightarrow e^{-x/2} = \boxed{\sum_{k=0}^{\infty} \frac{(-x/2)^k}{k!} = \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{2}\right)^k \frac{x^k}{k!} = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{2^k \cdot k!}}$$

6. $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

$$\Rightarrow e^{-x} = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}$$

$$x^2 e^{-x} = x^2 \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!} = \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{x^{k+2}}{k!}}$$

7. $\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

$$\Rightarrow \frac{\sin x}{x} = \frac{1}{x}\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!} = \boxed{\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k+1)!} = \frac{\sin x}{x}}$$

$$\int \frac{\sin x}{x}\,dx = \int \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k+1)!}\,dx = \sum_{k=0}^{\infty}\int (-1)^k \frac{x^{2k}}{(2k+1)!}\,dx$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\int x^{2k}\,dx = \boxed{\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\,\frac{x^{2k+1}}{2k+1} + C = \int \frac{\sin x}{x}\,dx}$$

8. $\sum_{n=0}^{\infty}\frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!} = \sum_{n=0}^{\infty}(-1)^n \frac{(\pi/6)^{2n}}{(2n)!} = \cos(\pi/6) = \boxed{\sqrt{3}/2}$

9. $\sum_{n=0}^{\infty}\frac{3^n}{5^n n!} = \sum_{n=0}^{\infty}\frac{(3/5)^n}{n!} = \boxed{e^{3/5}}$

10. If the series

$$1.6 - 0.8(x-1) + 0.4(x-1)^2 - 0.1(x-1)^3 + \cdots$$

is the Taylor series of f centered at $x=1$,

then $\frac{f'(1)}{1!} = -0.8$, so $f'(1) = -0.8$.

But f is increasing at $x=1$, so $f'(1)$ must be positive. Thus, $f'(1) \neq -0.8$.