Taylor Polynomials: Solutions to the Examples

Examples. See the posted Maple file for graphs.

1. Find the 5-th order Taylor (Maclaurin) polynomial P_5 for the function $f(x) = e^x$ centered at $x_0 = 0$.

   Solution. $P_5(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$. The key observation that you should make from the graph is that the accuracy of the approximation is best near the center $x_0 = 0$.

2. Find the Taylor (Maclaurin) polynomials P_1, P_3, P_5, and P_7 for the function $f(x) = \sin x$ centered at $x_0 = 0$. Plot P_1, P_3, P_5, P_7, and f on the same set of axes. Discuss the results.

   Solution.

$$\begin{aligned} P_1(x) &= x \\ P_3(x) &= x - \frac{1}{3!}x^3 \\ P_5(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \\ P_7(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 \end{aligned}$$

   The key observation that you should make from the graphs is that the accuracy of the approximations increases as the order n of the polynomial increases.

3. Find the 5-th order Taylor polynomial for the function $f(x) = \frac{1}{x}$ centered at $x_0 = 1$. Graph P_5 and f on the same set of axes.

   Solution. $P_5(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - (x-1)^5$

4. Find the 4-th order Taylor polynomial for the function $f(x) = \sqrt{x}$ centered at $x_0 = 9$. Graph P_4 and f on the same set of axes.

   Solution. $P_4(x) = 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2 + \frac{1}{3888}(x-9)^3 - \frac{5}{279936}(x-9)^4$. This polynomial provides a good approximation for the function $f(x) = \sqrt{x}$ around $x_0 = 9$. In particular, this polynomial could be used to approximate $\sqrt{10}$.

5. Find the 4-th order Taylor polynomial for the function $f(x) = \ln x$ centered at $x_0 = 4$. Graph P_4 and f on the same set of axes.

   Solution. $P_4(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$.