Taylor Polynomials

Recall that the Taylor series for a function $f(x)$ is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k,$$

where $f^{(k)}$ represents the k-th derivative of f. In practice (e.g. for graphing and numerical approximations), it is often difficult to work with infinite series representations of functions, so we instead approximate $f(x)$ with a partial sum

$$\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

This partial sum is called the **n-th order Taylor polynomial centered at x_0**:

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \\ &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \end{aligned}$$

The following are some important properties of Taylor polynomials:

- Taylor polynomials provide an *approximation* for a function $f(x)$.
- As with series, if the center of the polynomial is $x_0 = 0$, we call the resulting polynomial a **Maclaurin polynomial**:

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k \\ &= f(x_0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n. \end{aligned}$$

- A Taylor polynomial P_n of order n is constructed such that its value and first n derivatives at $x = x_0$ agree with those of $f(x)$.
- An n-th order Taylor polynomial is a polynomial of degree n.
- Taylor polynomials can be used to provide a *polynomial approximation* for more complicated functions. As the order of the Taylor polynomial increases, the approximation becomes a closer match to the actual function.

Examples.

1. Find the 5-th order Taylor (Maclaurin) polynomial P_5 for the function $f(x) = e^x$ centered at $x_0 = 0$.
2. Find the Taylor (Maclaurin) polynomials P_1, P_3, P_5, and P_7 for the function $f(x) = \sin x$ centered at $x_0 = 0$. Plot P_1, P_3, P_5, P_7, and f on the same set of axes. Discuss the results.
3. Find the 5-th order Taylor polynomial for the function $f(x) = \dfrac{1}{x}$ centered at $x_0 = 1$. Graph P_5 and f on the same set of axes.
4. Find the 4-th order Taylor polynomial for the function $f(x) = \sqrt{x}$ centered at $x_0 = 9$. Graph P_4 and f on the same set of axes.
5. Find the 4-th order Taylor polynomial for the function $f(x) = \ln x$ centered at $x_0 = 4$. Graph P_4 and f on the same set of axes.