Homework 12: Taylor and Maclaurin series.

solutions.

1. $f(x) = \cancel{\sin x}\ \cos x$ centered at $x = \cancel{\pi/16}\ \pi/2$.

$$\cancel{\sum_{k=0}^{\infty} \frac{f^{(k)}(\pi/16)}{k!}(x-\pi/16)^k}$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(\pi/2)}{k!}(x-\pi/2)^k$$

k	$f^{(k)}(x)$	$f^{(k)}(\pi/2)$
0	$\cos x$	0
1	$-\sin x$	-1
2	$-\cos x$	0
3	$\sin x$	1
4	$\cos x$	0

For even k, $f^{(k)}(\pi/2) = 0$.

For odd k, $f^{(k)}(\pi/2) = \pm 1$.

This cycle repeats.

The Taylor series is:

$$-1(x-\pi/2) + \frac{1}{3!}(x-\pi/2)^3 - \frac{1}{5!}(x-\pi/2)^5 + \cdots = \sum_{k=0}^{\infty} (-1)^{2k+1} \frac{(x-\pi/2)^{2k+1}}{(2k+1)!}$$

2. $\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

$$\sin(x^4) = \sum_{k=0}^{\infty} (-1)^k \frac{(x^4)^{2k+1}}{(2k+1)!} = \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{x^{8k+4}}{(2k+1)!}}$$

3. $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

$$e^{-3x} = \sum_{k=0}^{\infty} \frac{(-3x)^k}{k!} = \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k 3^k x^k}{k!}}$$

4. $e^{2x} = \sum_{k=0}^{\infty} \frac{(2x)^k}{k!} = \sum_{k=0}^{\infty} \frac{2^k x^k}{k!}$

$$xe^{2x} = x\sum_{k=0}^{\infty} \frac{2^k x^k}{k!} = \boxed{\sum_{k=0}^{\infty} \frac{2^k x^{k+1}}{k!}}$$

5. $\cos(x^2) = \sum_{k=0}^{\infty} (-1)^k \frac{(x^2)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k}}{(2k)!}$

$$x^3\cos(x^2) = x^3 \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k}}{(2k)!} = \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+3}}{(2k)!}}$$

6. $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\frac{e^x - 1}{x} = \boxed{1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots}$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{x^k}{(k+1)!}}$$

7. $\int \frac{e^x - 1}{x}\,dx = \int \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!}\,dx = \sum_{k=0}^{\infty} \int \frac{x^k}{(k+1)!}\,dx$

$$= \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \int x^k\,dx = \boxed{\sum_{k=0}^{\infty} \frac{1}{(k+1)!} \frac{x^{k+1}}{k+1} + C}$$

$$= \boxed{x + \frac{x^2}{2!\cdot 2} + \frac{x^3}{3!\cdot 3} + \cdots + C}$$

~~8.~~ $\cancel{\sum_{n=0}^{\infty} (-1)^n \frac{\pi^n}{9^n (2n)!} = \sum_{n=0}^{\infty} (-1)^n}$

8.

$$\sum_{n=0}^{\infty} (-1)^n \frac{(\pi/3)^{2n}}{(2n)!} = \cos(\pi/3) = \boxed{\frac{1}{2}}$$

9. $\sum_{n=0}^{\infty} \frac{(-e)^n}{n!} = \boxed{e^{-e}}$

10. If $2.8 + 0.5(x-2) + 1.5(x-2)^2 - 0.1(x-2)^3 + \cdots$

is the Taylor series of f centered at $x=2$,

then $\frac{f''(2)}{2!} = 1.5$, so $f''(2) = 3$. But f is

concave down at $x=2$, so $f''(2)$ must be

negative, so $f''(2) \neq 3$.