Taylor and Maclaurin Series

Suppose that we have a function $y = f(x)$, and that we want to find a power series representation for $f(x)$ centered at a point x_0:

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots .$$

$$\begin{aligned} f(x_0) &= a_0 \\ f'(x_0) &= a_1 \\ f''(x_0) &= 2a_2 \\ f'''(x_0) &= 3 \cdot 2a_3 \\ &\cdots \\ f^{(k)}(x_0) &= k!a_k \end{aligned}$$

Conclusion. The power series representation for a function $f(x)$ centered at x_0 (with infinitely many derivatives at x_0) is given by

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k,$$

where

$$a_k = \frac{f^{(k)}(x_0)}{k!},$$

where $f^{(k)}$ represents the k-th derivative of f. This power series is called the **Taylor series** for the function f centered at x_0. If $x_0 = 0$, then we call the resulting power series a **Maclaurin series**:

$$f(x) = \sum_{k=0}^{\infty} a_k x^k,$$

where

$$a_k = \frac{f^{(k)}(0)}{k!}.$$

These expressions give us a formula for constructing series representations for functions.

Examples.

1. Find the Maclaurin series for $f(x) = e^x$.
2. Find the Maclaurin series for $f(x) = e^{3x}$.
3. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^{4n}}{n!}$.
4. Find the Maclaurin series for $f(x) = \sin x$.
5. Find the sum of the series $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k+1}}{(2k+1)!}$.
6. Find the Maclaurin series for $f(x) = \cos x$.
7. Find the Maclaurin series for $f(x) = x \cos x$.
8. Find the Taylor series for $f(x) = \ln x$ centered at $x = 1$.