The Traveling Salesman Problem.

Given a graph with n vertices, having an edge between every pair of vertices, what is the shortest cycle which starts at a particular vertex, visits every other vertex exactly once, and returns to the original vertex?

1. One way to solve this problem is to simply enumerate all of the possible Hamilton cycles, determine the total length of each, and choose the shortest. For an n-city TSP, there are $n!$ total Hamilton cycles. So for n sufficiently small, a computer can solve the n-city TSP reasonably quickly. However, as n increases, the total computation time required to enumerate all $n!$ possibilities makes the brute force approach impossible. For example, it would take a 500 MHz computer approximately 7,715 years to solve the 20-city TSP.

2. The *Branch and Bound* method significantly reduces the total number of Hamilton cycles that must be checked.

   - In general, the lower bound for the TSP equals the sum of the constants subtracted from the rows and columns of the original cost matrix to obtain a new cost matrix with a 0 in each entry and column.
   - At any stage, as long as the lower bounds for partial tours using c_{ij} are less than the lower bound for tours not using c_{ij}, then we do not need to look at the subtree of possible tours not using c_{ij}.
   - At each stage, we should pick as the next entry on which to branch (use or do not use the entry)the 0 entry whose removal maximizes the increase in the lower bound.

3. The *TSP Quick Tour Construction* is a quicker algorithm for obtaining near-minimal tours when the costs are symmetric (i.e. $c_{ij} = c_{ji}$) and the costs satisfy the triangle inequality (i.e. $c_{ik} \leq c_{ij} + c_{jk}$). The Quick Tour is given by the following algorithm:

   (a) Pick any vertex as a starting circuit C_1 consisting of 1 vertex.

   (b) Given the k-vertex circuit C_k, $k \geq 1$, find the vertex z_k not on C_k that is closest to a vertex, call it y_k, on C_k.

   (c) Let C_{k+1} be the $k+1$-vertex circuit obtained by inserting z_k immediately in front of y_k in C_k.

   (d) Repeat the previous two steps until a Hamilton cycle is formed.

   Theorem. The cost of the tour generated by the Quick Tour construction is less than twice the cost of the minimal TSP tour.

Problems.

1. Solve the TSP using both the Branch and Bound method and the Quick Tour construction for the following cost matrix.

$$\begin{matrix} \infty & 3 & 3 & 2 & 7 & 3 \\ 3 & \infty & 3 & 4 & 5 & 5 \\ 3 & 3 & \infty & 1 & 4 & 4 \\ 2 & 4 & 1 & \infty & 5 & 5 \\ 7 & 5 & 4 & 5 & \infty & 4 \\ 3 & 5 & 4 & 5 & 4 & \infty \end{matrix}$$

2. Every month a plastics plant must make batches of five different types of plastic toys. There is a conversion cost c_{ij} in switching from the production of toy i to toy j, as shown in the following matrix. Find a sequence of toy production (be be followed for many months) that minimizes the sum of the monthly conversion costs.

$$\begin{matrix} \infty & 3 & 2 & 4 & 3 \\ 4 & \infty & 4 & 5 & 6 \\ 5 & 3 & \infty & 4 & 4 \\ 3 & 5 & 1 & \infty & 6 \\ 5 & 4 & 2 & 3 & \infty \end{matrix}$$

3. Find a 3×3 cost matrix for which two different initial lower bounds can be obtained (with different sets of 0 entries) by subtracting from the rows and columns in different orders.

4. Make up a 5×5 cost matrix for which the Quick Tour construction finds:

   (a) An optimal tour.

   (b) A fairly costly tour (at least 50% over the true minimum).