Math 333
Tuesday, February 5, 2008
Systems of First-Order Differential Equations

A **system of first-order differential equations** is a collection of two (or more) differential equations of the form:

$$\frac{dx}{dt} = f(t, x, y)$$
$$\frac{dy}{dt} = g(t, x, y).$$

The solution of a system of first-order DE's is a pair of functions $(x(t), y(t))$ that satisfies the differential equations above.
We'll consider autonomous systems of the form:

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y).$$

The classical example of a situation in which systems of first-order DE's naturally arise is modeling of **predator-prey** systems. There are two classic models that we'll consider. For both models, we'll use the following notation:

- $x(t)$ denotes the prey population at time t
- $y(t)$ denotes the predator population at time t
- $\alpha_x > 0$ denotes the natural growth rate of the prey population (in the absence of predators)
- $\alpha_y > 0$ denotes the natural death rate of the predator population (in the absence of prey)
- $\gamma_x > 0$ is a parameter that measures the number of predator-prey interactions in which a prey is eaten
- $\gamma_y > 0$ is a parameter that measures the benefit to the predator population of eating prey
- N_x denotes the maximum sustainable prey population (in the absence of predators)

Model 1: Assume that the prey population follows an **exponential** growth model in the absence of predators. Then the predator-prey system that models the two populations is:

Model 2: Assume that the prey population follows a **logistic** growth model in the absence of predators. Then the predator-prey system that models the two populations is:

For now, we'll use **graphical techniques** to analyze systems of first-order differential equations. There are four main types of graphs that will be useful in analyzing systems of DE's:

- x as a function of t
- y as a function of t
- The phase portrait or phase plane:
- The direction field:

Example: Analyze the predator-prey type system

$$\begin{aligned}\frac{dx}{dt} &= 2x - xy \\ \frac{dy}{dt} &= -3y + xy.\end{aligned}$$

- Plot $x(t)$ and $y(t)$ for various initial conditions. Interpret the results in terms of the predator-prey population system.
- Draw the phase portrait in the xy-plane for various initial conditions. Interpret the results in terms of the predator-prey population system.
- Draw the direction field in the xy-plane.
- Find the equilibrium points of the system, and discuss the phase portrait and direction field near each equilibrium point.