Series Solutions Near a Regular Singular Point

We will now consider solving the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{1}$$

in the neighborhood of a regular singular point x_0. For convenience, we'll assume that $x_0 = 0$. If $x_0 \neq 0$, we can transform the equation into one for which the regular singular point is 0 by making the change of variables $t = x - x_0$. The key steps in constructing solutions of Eqn. (1) in the neighborhood of a regular singular point $x_0 = 0$ are the following.

1. First, divide both sides of Eqn. (1) by $P(x)$ to obtain:

$$y'' + p(x)y' + q(x)y = 0, \tag{2}$$

   where $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$.

2. The fact that $x_0 = 0$ is a *regular* singular point of Eqn. (1) means that $\lim\limits_{x\to 0} \dfrac{xQ(x)}{P(x)} = \lim\limits_{x\to 0} xp(x)$ is finite and $\lim\limits_{x\to 0} \dfrac{x^2R(x)}{P(x)} = \lim\limits_{x\to 0} x^2q(x)$ is finite. Thus, $xp(x)$ and $x^2q(x)$ have convergent power series centered at $x_0 = 0$:

   - $xp(x) = \sum\limits_{n=0}^{\infty} p_n x^n$
   - $x^2q(x) = \sum\limits_{n=0}^{\infty} q_n x^n$

3. To make the quantities $xp(x)$ and $x^2q(x)$ appear in Eqn. (2), we multiply both sides by x^2 to obtain:

$$x^2y'' + x[xp(x)]y' + [x^2q(x)]y = 0. \tag{3}$$

4. Using the power series expansions for $xp(x)$ and $x^2q(x)$ centered at $x_0 = 0$, we can rewrite Eqn. (3) as:

$$x^2y'' + x[p_0 + p_1x + p_2x^2 + \cdots]y' + [q_0 + q_1x + q_2x^2 + \cdots]y = 0. \tag{4}$$

5. If all of the coefficients p_n and q_n are zero, except possibly

$$p_0 = \lim_{x\to 0} xp(x)$$

   and

$$q_0 = \lim_{x\to 0} x^2q(x),$$

   then Eqn. (4) reduces to the Euler equation

$$x^2y'' + p_0xy' + q_0y = 0, \tag{5}$$

which we have discussed in detail previously. We call Eqn. (5) the **Euler equation corresponding to Eqn. (1)**.

6. In general, of course, some of the p_n and q_n will be non-zero. However, the essential character of solutions of Eqn. (4) will be the same as that of solutions of the Euler equation (5). The presence of the terms

$$p_1x + p_2x^2 + \cdots \text{ and } q_1x + q_2x^2 + \cdots$$

just complicates our calculations.

7. For now, we'll restrict our discussion to the interval $x > 0$. The interval $x < 0$ can be treated in the same way as for the Euler equation by making the change of variable $x = -\gamma$ and then solving the resulting equation for $\gamma > 0$.

8. The key observation now is that since the coefficients in Eqn. (4) are "Euler coefficients" times power series, it is natural to guess solutions in the form of "Euler solutions" times power series. Thus, we look for a solution of the form

$$y = x^r(a_0 + a_1x + a_2x^2 + \cdots) = x^r \sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty} a_nx^{r+n}. \tag{6}$$

Thus, r is the exponent of the first term in the series, and a_0 is its coefficient.

9. To determine the solution, we must find the following:

   (a) The values of r for which Eqn. (1) has a solution of the form in (6).
   (b) The recurrence relation for the coefficients a_n.
   (c) The radius of convergence of the series.

The general theory was constructed by Ferdinand Georg Frobenius in 1874. Rather than trying to present the general theory here (which is rather complicated), we'll start with an example.

Example. Solve the differential equation

$$2x^2y'' - xy' + (1 + x)y = 0. \tag{7}$$

First, note that $x = 0$ is a regular singular point of Eqn. (7). Rewriting Eqn. (7) in the form $y'' + p(x)y' + q(x)y = 0$, we obtain

$$y'' - \frac{1}{2x}y' + \frac{(1 + x)}{2x^2}y = 0.$$

Thus,

$$xp(x) = \frac{-1}{2} \text{ and } x^2q(x) = \frac{1+x}{2} = \frac{1}{2} + \frac{x}{2}.$$

Thus:

$$\begin{aligned} p_0 &= -1/2 \\ q_0 &= 1/2 \\ q_1 &= 1/2 \end{aligned}$$

All other p_n and q_n are equal to zero. Note that the Euler equation corresponding to Eqn. (7) is then

$$x^2y'' - \frac{1}{2}xy' + \frac{1}{2}y = 0. \tag{8}$$

To solve Eqn. (7), we assume that there is a solution of the form (6), and compute y' and y'':

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{r+n} \\ y' &= \sum_{n=0}^{\infty} a_n (r+n) x^{r+n-1} \\ y'' &= \sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n-2} \end{aligned}$$

Substituting y, y', and y'' in Eqn. (7), we obtain:

$$\begin{aligned} 2x^2y'' - xy' + (1+x)y &= \sum_{n=0}^{\infty} 2a_n(r+n)(r+n-1)x^{r+n} \\ &\quad - \sum_{n=0}^{\infty} a_n(r+n)x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+1} \\ &= a_0[2r(r-1) - r + 1]x^r \\ &\quad + \sum_{n=1}^{\infty} \left[[2(r+n)(r+n-1) - (r+n) + 1]a_n + a_{n-1}\right] x^{r+n} \\ &= 0. \end{aligned}$$

Next, we equate coefficients. From the coefficient of x^r, we obtain

$$2r(r-1) - r + 1 = (r-1)(2r-1) = 0. \tag{9}$$

Eqn. (9) is called the **indicial equation** for Eqn. (7). Note that it is exactly the same polynomial equation that we would obtain for the Euler equation (8) associated with Eqn. (7). The roots of the indicial equation are

$$r_1 = 1, \quad r_2 = 1/2.$$

These values of r are called the **exponents** at the singularity for the regular singular point $x_0 = 0$. They determine the behavior of the solution in the neighborhood of the singular point.

Next, we set the coefficient of x^{r+n} equal to 0. This gives the relation

$$[2(r+n)(r+n-1) - (r+n) + 1]a_n + a_{n-1} = 0,$$

or

$$a_n = -\frac{a_{n-1}}{2(r+n)^2 - 3(r+n) + 1}, \quad n \geq 1.$$

For each root r_1 and r_2 of the indicial equation, we use this recurrence relation to determine a set of coefficients $a_1,\ a_2, \ldots$. For $r = r_1 = 1$, the recurrence relation becomes

$$a_n = -\frac{a_{n-1}}{(2n+1)n},$$

and we obtain:

$$\begin{aligned}
a_1 &= -\frac{a_0}{3 \cdot 1} \\
a_2 &= -\frac{a_1}{5 \cdot 2} = \frac{a_0}{(3 \cdot 5)(1 \cdot 2)} \\
a_3 &= -\frac{a_2}{7 \cdot 3} = -\frac{a_0}{(3 \cdot 5 \cdot 7)(1 \cdot 2 \cdot 3)} \\
a_n &= \frac{(-1)^n a_0}{[3 \cdot 5 \cdot 7 \cdots (2n+1)]n!} \\
&= \frac{(-1)^n 2^n a_0}{(2n+1)!}
\end{aligned}$$

To get the last line, we multiply the numerator and denominator of the previous line by $2 \cdot 4 \cdot 6 \cdots 2n = 2^n n!$. Thus, one solution of Eqn.(7) is given by:

$$y_1(x) = x^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(2n+1)!} x^n \right], \quad x > 0$$

where we have omitted the arbitrary constant multiplier a_0. We can use the ratio test to show that this series converges for all x.

We proceed similarly for the second root $r = r_2 = \frac{1}{2}$. The recurrence relation becomes

$$a_n = -\frac{a_{n-1}}{n(2n-1)},$$

and we obtain:

$$\begin{aligned}
a_1 &= -\frac{a_0}{1\cdot 1} \\
a_2 &= -\frac{a_1}{2\cdot 3} = \frac{a_0}{(1\cdot 2)(1\cdot 3)} \\
a_3 &= -\frac{a_2}{3\cdot 5} = -\frac{a_0}{(1\cdot 2\cdot 3)(1\cdot 3\cdot 5)} \\
a_n &= \frac{(-1)^n a_0}{n![1\cdot 3\cdot 5\cdot 7\cdots(2n-1)]} \\
&= \frac{(-1)^n 2^n a_0}{(2n)!}
\end{aligned}$$

To get the last line, we multiply the numerator and denominator of the previous line by $2^n n!$. Thus, again omitting the constant multiplier a_0,we obtain the second solution:

$$y_2(x) = x^{1/2}\left[\sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{(2n!)}x^n\right], \quad x > 0.$$

As before, we can use the Ratio Test to show that the series converges for all x. Since the leading terms in the series solutions y_1 and y_2 are x and $x^{1/2}$, respectively, the solutions are linearly independent. Thus, the general solution of Eqn. (7) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x), \quad x > 0.$$

This example illustrates that if $x = 0$ is a regular singular point, then sometimes there are two solutions of the form (6) in the neighborhood of this point. Similarly, if there is a regular singular point at $x = x_0$, then there may be two solutions of the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{10}$$

that are valid near $x = x_0$. However, just as an Euler equation may not have two solutions of the form $y = x^r$, so a more general equation with a regular singular point may not have two solutions of the form (6) or (10). In particular, if the roots r_1 and r_2 are equal, or differ by an integer, then the second solution has a more complicated structure. In all cases, though, it is possible to find at least one solution of the form (6) or (10); if r_1 and r_2 differ by an integer, then this solution corresponds to the larger value of r. If there is only one such solution, then the second solution involves a logarithmic term, just as for the Euler equation when the roots of the characteristic equation are equal. The method of reduction of order can be used to determine the second solution in such cases. If the roots of the indicial equation are complex, then they cannot be equal or differ by an integer, so there are always two solutions of the form (6) or (10) that we can express as real-valued solutions.