Math 333

Homework: Sinusoidal Forcing

Solutions

1. $y'' + y = 3\cos(\omega t)$, $y(0) = 0$, $y'(0) = 0$.

(a) $y'' + y = 0 \Rightarrow r^2 + 1 = 0 \Rightarrow r^2 = -1 \quad \lambda = 0, \mu = 1$

$y_h(t) = k_1 \cos t + k_2 \sin t$

$y_p(t) = A\sin\omega t + B\cos\omega t$

$y_p' = A\omega\cos\omega t - B\omega\sin\omega t \quad y_p'' = -A\omega^2\sin\omega t - B\omega^2\cos\omega t$

$y_p'' + y_p = 3\cos(\omega t)$

$(-A\omega^2 + A)\sin(\omega t) + (-B\omega^2 + B)\cos(\omega t) = 3\cos(\omega t)$

$-A\omega^2 + A = 0 \qquad -B\omega^2 + B = 3$

$A = 0 \qquad B = \dfrac{3}{1-\omega^2}, \ \omega \neq 1.$

For $\omega \neq 1$, $y_p(t) = \dfrac{3}{1-\omega^2}\cos(\omega t)$

$\rightarrow$

So for $\omega \neq 1$, the general solution is

$$y(t) = K_1 \cos t + K_2 \sin t + \frac{3}{1-\omega^2} \cos(\omega t)$$

$$y(0) = 0 \Rightarrow K_1 + \frac{3}{1-\omega^2} = 0 \Rightarrow K_1 = \frac{-3}{1-\omega^2}$$

$$y'(0) = 0 \Rightarrow K_2 = 0.$$

$$\boxed{\omega \neq 1: y(t) = -\frac{3}{1-\omega^2} \cos t + \frac{3}{1-\omega^2} \cos(\omega t)}$$

$\omega = 1$: $y_p(t) = At\sin(\omega t) + Bt\cos(\omega t)$

$$y_p'' + y_p = 3\cos(\omega t) \Rightarrow A = \frac{3}{2}$$

$$y_p(t) = \frac{3}{2} t \sin(\omega t).$$

So for $\omega = 1$, the general solution is

$$y(t) = K_1 \cos t + K_2 \sin t + \frac{3}{2} t \sin(\omega t)$$

$y(0)=0 \Rightarrow K_1=0$

$y'(0)=0 \Rightarrow K_2=0$

$\boxed{\underline{w=1}: y(t)=\frac{3}{2}t\sin(\omega t)}$

The natural frequency of the oscillator is $\frac{1}{2\pi}$.

(b) See Maple plots.

As $\omega \to 1$, beating occurs. As ω gets closer to 1, the period and amplitude of the beats increases.

When $\omega=1$, the system is in resonance, and the amplitude of oscillations increases linearly with time.

$\rightarrow$

2. See Maple. To graphically simulate the DE, first convert it to a system of first order DE's.

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = \cos(\omega t) - \frac{1}{5}v - y - \frac{1}{5}y^3$$

(b) The amplitude appears greatest for $\omega = 1$.

(c) $\frac{dy}{dt} = v \quad \frac{dv}{dt} = -\frac{1}{5}v - y + \cos(\omega t)$