

Math 333

Homework: Sinusoidal Forcing

Solutions

1. $y'' + y = 3\cos(\omega t)$, $y(0) = 0$, $y'(0) = 0$.

(a) $y'' + y = 0 \Rightarrow r^2 + 1 = 0 \Rightarrow r^2 = -1 \Rightarrow \lambda = 0, \mu = 1$

$$y_h(t) = K_1 \cos t + K_2 \sin t$$

$$y_p(t) = A \sin \omega t + B \cos \omega t$$

$$y_p' = Aw \cos \omega t - Bw \sin \omega t \quad y_p'' = -Aw^2 \sin \omega t - Bw^2 \cos \omega t$$

$$y_p'' + y_p = 3 \cos(\omega t)$$

$$(-Aw^2 + A) \sin(\omega t) + (-Bw^2 + B) \cos(\omega t) = 3 \cos(\omega t)$$

$$-Aw^2 + A = 0 \quad -Bw^2 + B = 3$$

$$A = 0 \quad B = \frac{3}{1-w^2}, \quad w \neq 1.$$

For $w \neq 1$, $y_p(t) = \frac{3}{1-w^2} \cos(\omega t)$



so for $\omega \neq 1$, the general solution is

$$y(t) = K_1 \cos t + K_2 \sin t + \frac{3}{1-\omega^2} \cos(\omega t)$$

$$y(0) = 0 \Rightarrow K_1 + \frac{3}{1-\omega^2} = 0 \Rightarrow K_1 = -\frac{3}{1-\omega^2}$$

$$y'(0) = 0 \Rightarrow K_2 = 0.$$

$$\boxed{\omega \neq 1: y(t) = -\frac{3}{1-\omega^2} \cos t + \frac{3}{1-\omega^2} \cos(\omega t)}$$

$$\omega = 1: y_p(t) = At \sin(\omega t) + Bt \cos(\omega t)$$

$$y_p'' + y_p = 3 \cos(\omega t) \Rightarrow A = \frac{3}{2}$$

$$y_p(t) = \frac{3}{2} t \sin(\omega t).$$

so for $\omega = 1$, the general solution is

$$y(t) = K_1 \cos t + K_2 \sin t + \frac{3}{2} t \sin(\omega t)$$

$$y(0)=0 \Rightarrow k_1=0$$

$$y'(0)=0 \Rightarrow k_2=0$$

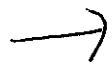
$$\boxed{w=1: y(t)=\frac{3}{2} t \sin(\omega t)}$$

The natural frequency of the oscillator is $\frac{1}{2\pi}$.

(b) See Maple plots.

As $w \rightarrow 1$, beating occurs. As $w \neq$ gets closer to 1, the period and amplitude of the beats increases.

When $w=1$, the system is in resonance, and the amplitude of oscillations increases linearly with time.



2. See Maple. To graphically simulate the DE,
first convert it to a system of first-order DE's.

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = \cos(\omega t) - \frac{1}{5}v - y - \frac{1}{5}y^3$$

(b) The amplitude appears greatest for $\omega=1$.

(c) $\frac{dy}{dt} = v \quad \frac{dv}{dt} = -\frac{1}{5}v - y + \cos(\omega t)$