Math 333
Tuesday, April 1, 2008
Qualitative Results: Sinusoidal Forcing and Resonance

Note: The material presented in these lecture notes corresponds (approximately) to Sections 4.2 and 4.3 in your textbook. Section 4.2 presents an alternative method for solving second-order linear non-homogeneous differential equations when the forcing function $g(t)$ is a sine or cosine function. We will discuss the method of undetermined coefficients in class, but you are welcome to read Section 4.2 for an alternative method. You will not be responsible (on quizzes, homeworks, or tests) for the complexification method presented in Section 4.2, though you are welcome to use the complexification method if you prefer it over the undetermined coefficients method.

Consider the forced harmonic oscillator equation

$$y'' + py' + qy = g(t),$$

where p and q are positive constants and the forcing function $g(t)$ is a sine or cosine function. This type of external forcing function occurs frequently in applications. We can solve such equations using the Extended Linearity Principle and the Method of Undetermined Coefficients. We have seen previously that all solutions approach the steady-state response. In applications, we'll often be interested in determining the *amplitude* of the steady-state response. We'll need the following trigonometric identity:

$$A \sin x + B \cos x = C \sin(x + \varphi),$$

where

$$C = \sqrt{A^2 + B^2} \text{ and } \varphi = \tan^{-1}(\frac{B}{A}).$$

Example 1: Sinusoidal Forcing of a Damped Harmonic Oscillator. Consider the forced damped harmonic oscillator equation

$$y'' + 3y' + 2y = \cos t.$$

Find the general solution of the equation, and discuss the long-term behavior of solutions of the equation. In particular, determine the amplitude of the steady-state response.

Solution. The general solution of the differential equation is

$$y(t) = k_1 e^{-t} + k_2 e^{-2t} + \frac{\sqrt{10}}{10} \sin(t + \arctan(1/3)).$$

As $t \to \infty$, $y(t)$ approaches the steady-state response

$$y_p(t) = \frac{\sqrt{10}}{10} \sin(t + \arctan(1/3)).$$

The amplitude of the steady-state response is $\frac{\sqrt{10}}{10}$ and the frequency is $1/2\pi$.

Next, we will consider sinusoidal forcing of *undamped* equations. We will be particularly interested in a qualitative analysis of how solutions depend on the *frequency* of the forcing term. Recall that the **period** of

$$\cos(\omega t)$$

is

$$\mathrm{T} = \frac{2\pi}{\omega}$$

and the **frequency** is

$$f = \frac{\omega}{2\pi}.$$

Example 2: Sinusoidal Forcing of an Undamped Harmonic Oscillator.
Consider the forced undamped harmonic oscillator equation

$$y'' + 2y = \cos(\omega t),$$

where ω is a numerical parameter that determines the frequency of the external forcing: $f = \frac{\omega}{2\pi}$. This is called the **forcing frequency**.

1. Find the general solution of this differential equation. Note that we will need to consider the cases $\omega \neq \sqrt{2}$ and $\omega = \sqrt{2}$ separately.

2. Graph the particular solutions with initial conditions $y(0) = 0, \quad y'(0) = 0$ for $\omega = 0.5$, $\omega = 1$, $\omega = 1.2$, $\omega = 1.4$, and $\omega = 1.41$. Discuss the qualitative behavior of these solutions. In particular, discuss the dependence of the amplitude of the forced response on the frequency of the forcing.

3. Graph the particular solution with initial conditions $y(0) = 0, \quad y'(0) = 0$ for $\omega = \sqrt{2}$. Discuss the results.

Solution. The general solution of the associated homogeneous equation is

$$y_h(t) = k_1 \cos(\sqrt{2}t) + k_2 \sin(\sqrt{2}t).$$

The frequency of the natural response is $\frac{\sqrt{2}}{2\pi}$. This is called the **natural frequency** of the oscillator. For $\omega \neq \sqrt{2}$, the general solution of the forced equation is

$$y(t) = k_1 \cos(\sqrt{2}t) + k_2 \sin(\sqrt{2}t) + \frac{1}{2 - \omega^2} \cos(\omega t).$$

If $\omega = \sqrt{2}$, then the frequency of the external force is exactly the same as the natural frequency. Forcing whose frequency is the same as the natural frequency of the oscillator is called **resonant frequency**, and the oscillator is said to be in **resonance**. For $\omega = \sqrt{2}$, the general solution is

$$y(t) = k_1 \cos(\sqrt{2}t) + k_2 \sin(\sqrt{2}t) + \frac{1}{2\sqrt{2}} t \sin(\sqrt{2}t).$$

The structure observed in the graph of the solution for $\omega = 1.2$ is called **beating**. Beating occurs when the forcing frequency is very close to the natural frequency. Beating is characterized by solutions in which the amplitudes increase and decrease in a regular pattern. As $\omega/2\pi$ becomes closer and closer to the natural frequency, the period (length) of the beats increases, and the amplitude of the oscillations increases. When $\omega = \sqrt{2}$ is exactly equal to the natural frequency, the oscillator is in resonance and the amplitude of the oscillations increases linearly with time.

Summary. To summarize, consider the undamped forced harmonic oscillator

$$y'' + qy = g(t),$$

where $q > 0$ and $g(t)$ is a sine or cosine function with frequency $\omega/2\pi$. Beating occurs when the forcing frequency $\omega/2\pi$ is close to the natural frequency of the oscillator. As $\omega/2\pi$ becomes closer and closer to the natural frequency, the lengths of the beats increases, and the amplitude of the beats increases. When the forcing frequency is exactly equal to the natural frequency, the oscillator is in resonance, and the amplitude of oscillations increases linearly with time. Resonance is an important consideration in the design of structures such as bridges and buildings, where it can produce instabilities that might lead to catastrophic failure of the structure.