Tests for Convergence and Divergence of Series

- **The n-th term test for divergence.** If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=0}^{\infty} a_n$ diverges.
- **Geometric series.** The geometric series

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots$$

  - converges to $\dfrac{a}{1-r}$ if $|r| < 1$
  - diverges if $|r| \geq 1$

- **Integral test for positive series.** Suppose that, for all $x \geq 1$, the function $a(x)$ is continuous, positive, and decreasing. Let $a_k = a(k)$ for all integers $k \geq 1$. Consider the series and the integral

$$\sum_{k=1}^{\infty} a_k \text{ and } \int_1^{\infty} a(x)\,dx.$$

  - If either diverges, so does the other.
  - If either converges, so does the other.

- **The p-test for series.** The p-series $\sum_{k=1}^{\infty} \dfrac{1}{k^p}$
  - converges if $p > 1$.
  - diverges if $p \leq 1$

- **Comparison test for non-negative series.** Consider two series Σa_k and Σb_k. Suppose that

$$0 \leq a_k \leq b_k$$

  for all k.

  - If Σb_k converges, so does Σa_k.
  - If Σa_k diverges, so does Σb_k.

- **Ratio test.** Let
$$L = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|.$$
  - If $L < 1$, then Σa_k converges absolutely, and therefore converges.
  - If $L > 1$, then Σa_k diverges.
  - If $L = 1$, then the test gives no information, and other techniques must be used.

- **Alternating Series Test.** If the alternating series
$$\sum_{k=1}^{\infty} (-1)^{k+1} c_k$$
satisfies
  1. $c_{k+1} \leq c_k$ for all k, and
  2. $\lim_{k\to\infty} c_k = 0$,

  then the series converges.