# Practice Problems: Series Solutions of differential equations near an ordinary point.

**PROBLEMS**

In each of Problems 1 through 14 solve the given differential equation by means of a power series about the given point $x_0$. Find the recurrence relation; also find the first four terms in each of two linearly independent solutions (unless the series terminates sooner). If possible, find the general term in each solution.

1. $y'' - y = 0, \quad x_0 = 0$
2. $y'' - xy' - y = 0, \quad x_0 = 0$
3. $y'' - xy' - y = 0, \quad x_0 = 1$
4. $y'' + k^2x^2y = 0, \quad x_0 = 0$, $k$ a constant
5. $(1-x)y'' + y = 0, \quad x_0 = 0$
6. $(2+x^2)y'' - xy' + 4y = 0, \quad x_0 = 0$
7. $y'' + xy' + 2y = 0, \quad x_0 = 0$
8. $xy'' + y' + xy = 0, \quad x_0 = 1$
9. $(1+x^2)y'' - 4xy' + 6y = 0, \quad x_0 = 0$
10. $(4-x^2)y'' + 2y = 0, \quad x_0 = 0$
11. $(3-x^2)y'' - 3xy' - y = 0, \quad x_0 = 0$
12. $(1-x)y'' + xy' - y = 0, \quad x_0 = 0$
13. $2y'' + xy' + 3y = 0, \quad x_0 = 0$
14. $2y'' + (x+1)y' + 3y = 0, \quad x_0 = 2$

# Answers to the practice problems:

1. $a_{n+2} = a_n/(n+2)(n+1)$

$$y_1(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \cosh x$$

$$y_2(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sinh x$$

2. $a_{n+2} = a_n/(n+2)$

$$y_1(x) = 1 + \frac{x^2}{2} + \frac{x^4}{2\cdot 4} + \frac{x^6}{2\cdot 4\cdot 6} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$$

$$y_2(x) = x + \frac{x^3}{3} + \frac{x^5}{3\cdot 5} + \frac{x^7}{3\cdot 5\cdot 7} + \cdots = \sum_{n=0}^{\infty} \frac{2^n n! x^{2n+1}}{(2n+1)!}$$

3. $(n+2)a_{n+2} - a_{n+1} - a_n = 0$

$y_1(x) = 1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \cdots$

$y_2(x) = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \cdots$

4. $a_{n+4} = -k^2a_n/(n+4)(n+3); \quad a_2 = a_3 = 0$

$$y_1(x) = 1 - \frac{k^2x^4}{3\cdot 4} + \frac{k^4x^8}{3\cdot 4\cdot 7\cdot 8} - \frac{k^6x^{12}}{3\cdot 4\cdot 7\cdot 8\cdot 11\cdot 12} + \cdots$$

$$= 1 + \sum_{m=0}^{\infty} \frac{(-1)^{m+1}(k^2x^4)^{m+1}}{3\cdot 4\cdot 7\cdot 8\cdots(4m+3)(4m+4)}$$

$$y_2(x) = x - \frac{k^2x^5}{4\cdot 5} + \frac{k^4x^9}{4\cdot 5\cdot 8\cdot 9} - \frac{k^6x^{13}}{4\cdot 5\cdot 8\cdot 9\cdot 12\cdot 13} + \cdots$$

$$= x\left[1 + \sum_{m=0}^{\infty} \frac{(-1)^{m+1}(k^2x^4)^{m+1}}{4\cdot 5\cdot 8\cdot 9\cdots(4m+4)(4m+5)}\right]$$

*Hint:* Let $n = 4m$ in the recurrence relation, $m = 1, 2, 3, \ldots$.

5. $(n+2)(n+1)a_{n+2} - n(n+1)a_{n+1} + a_n = 0, \quad n \geq 1; \quad a_2 = -\frac{1}{2}a_0$
$y_1(x) = 1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 + \cdots, \quad y_2(x) = x - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{1}{24}x^5 + \cdots$

6. $a_{n+2} = -(n^2 - 2n + 4)a_n/[2(n+1)(n+2)], \quad n \geq 2; \quad a_2 = -a_0, \quad a_3 = -\frac{1}{4}a_1$
$y_1(x) = 1 - x^2 + \frac{1}{6}x^4 - \frac{1}{30}x^6 + \cdots, \qquad y_2(x) = x - \frac{1}{4}x^3 + \frac{7}{160}x^5 - \frac{19}{1920}x^7 + \cdots$

7. $a_{n+2} = -a_n/(n+1), \quad n = 0, 1, 2, \ldots$

$$y_1(x) = 1 - \frac{x^2}{1} + \frac{x^4}{1\cdot 3} - \frac{x^6}{1\cdot 3\cdot 5} + \cdots = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{1\cdot 3\cdot 5\cdots(2n-1)}$$

$$y_2(x) = x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} + \cdots = x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2\cdot 4\cdot 6\cdots(2n)}$$

8. $a_{n+2} = -[(n+1)^2 a_{n+1} + a_n + a_{n-1}]/(n+1)(n+2), \quad n = 1, 2, \ldots$
$a_2 = -(a_0 + a_1)/2$
$y_1(x) = 1 - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{12}(x-1)^4 + \cdots$
$y_2(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{6}(x-1)^4 + \cdots$

9. $(n+2)(n+1)a_{n+2} + (n-2)(n-3)a_n = 0; \quad n = 0, 1, 2, \ldots$
$y_1(x) = 1 - 3x^2, \quad y_2(x) = x - x^3/3$

---

10. $4(n+2)a_{n+2} - (n-2)a_n = 0; \quad n = 0, 1, 2, \ldots$

$$y_1(x) = 1 - \frac{x^2}{4}, \quad y_2(x) = x - \frac{x^3}{12} - \frac{x^5}{240} - \frac{x^7}{2240} - \cdots - \frac{x^{2n+1}}{4^n(2n-1)(2n+1)} - \cdots$$

11. $3(n+2)a_{n+2} - (n+1)a_n = 0; \quad n = 0, 1, 2, \ldots$

$$y_1(x) = 1 + \frac{x^2}{6} + \frac{x^4}{24} + \frac{5}{432}x^6 + \cdots + \frac{3\cdot 5\cdots(2n-1)}{3^n\cdot 2\cdot 4\cdots(2n)}x^{2n} + \cdots$$

$$y_2(x) = x + \frac{2}{9}x^3 + \frac{8}{135}x^5 + \frac{16}{945}x^7 + \cdots + \frac{2\cdot 4\cdots(2n)}{3^n\cdot 3\cdot 5\cdots(2n+1)}x^{2n+1} + \cdots$$

12. $(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + (n-1)a_n = 0; \quad n = 0, 1, 2, \ldots$

$$y_1(x) = 1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots + \frac{x^n}{n!} + \cdots, \quad y_2(x) = x$$

13. $2(n+2)(n+1)a_{n+2} + (n+3)a_n = 0; \quad n = 0, 1, 2, \ldots$

$$y_1(x) = 1 - \frac{3}{4}x^2 + \frac{5}{32}x^4 - \frac{7}{384}x^6 + \cdots + (-1)^n\frac{3\cdot 5\cdots(2n+1)}{2^n(2n)!}x^{2n} + \cdots$$

$$y_2(x) = x - \frac{x^3}{3} + \frac{x^5}{20} - \frac{x^7}{210} + \cdots + (-1)^n\frac{4\cdot 6\cdots(2n+2)}{2^n(2n+1)!}x^{2n+1} + \cdots$$

14. $2(n+2)(n+1)a_{n+2} + 3(n+1)a_{n+1} + (n+3)a_n = 0; \quad n = 0, 1, 2, \ldots$
$y_1(x) = 1 - \frac{3}{4}(x-2)^2 + \frac{3}{8}(x-2)^3 + \frac{1}{64}(x-2)^4 + \cdots$
$y_2(x) = (x-2) - \frac{3}{4}(x-2)^2 + \frac{1}{24}(x-2)^3 + \frac{9}{64}(x-2)^4 + \cdots$

---