Math 333
Outline: Series Solutions of Differential Equations

The general procedure for finding a series solution of a second-order linear homogeneous differential equations of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{1}$$

is the following:

1. Guess a series solution of Eqn. (1) of the form

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

   around an *ordinary point* x_0. An ordinary point is any point x_0 such that $P(x_0) \neq 0$.

2. Compute y' and y'' as follows:

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} na_n(x - x_0)^{n-1} \\ &= \sum_{n=0}^{\infty} (n+1)a_{n+1}(x - x_0)^n \\ &= a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots \\ y'' &= \sum_{n=2}^{\infty} n(n-1)a_n(x - x_0)^{n-2} \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x - x_0)^n \\ &= 2a_2 + 3 \cdot 2a_3(x - x_0) + \cdots \end{aligned}$$

3. Substitute y, y', and y'' into Eqn. (1) to determine the coefficients a_n so that the function $y(x)$ actually satisfies the differential equation. This will usually require finding one or more recurrence relations for the coefficients a_n. Remember that $a_0 = y(0)$ and $a_1 = y'(0)$, so a_0 and a_1 are arbitrary constants.