Math 333
Series Solutions Near an Ordinary Point

We now consider methods of solving second order homogeneous linear differential equations when the coefficients are functions of the independent variable:

$$P(x)y'' + Q(x)y' + R(x)y = 0. \tag{1}$$

We will primarily consider the case in which the functions P, Q, and R are polynomials. However, the method of solution that we will develop is also applicable when P, Q, and R are general analytic functions. For now, suppose that P, Q, and R are polynomials and that they have no common factors. Suppose also that we wish to solve Eqn. (1) in a neighborhood of some point x_0. The solution of Eqn. (1) in an interval containing x_0 is closely associated with the behavior of P in that interval.

A point x_0 such that $P(x_0) \neq 0$ is called an **ordinary point**. Since P is continuous, it follow that there is an interval about x_0 in which $P(x)$ is never zero. In that interval, we can divide Eqn. (1) by $P(x)$ to obtain

$$y'' + p(x)y' + q(x)y = 0, \tag{2}$$

where $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$ are continuous functions. Hence, according to the Existence and Uniqueness Theorem that we have studied previously, there exists in that interval a unique solution of Eqn. (1) that also satisfies the initial conditions $y(x_0) = y_0$ and $y'(x_0) = y'_0$.

If $P(x_0) = 0$, then x_0 is called a **singular point** of x_0, and at least one of the coefficients p and q in Eqn. (2) becomes unbounded as $x \to x_0$, so the Existence and Uniqueness Theorem does not apply.

For now, we'll consider the problem of solving Eqn. (1) in the neighborhood of an ordinary point x_0. We'll return later to the problem of finding solutions of Eqn. (1) in the neighborhood of a singular point.

We look for solutions of Eqn. (1) of the form

$$y = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + \cdots = \sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

and assume that the series converges in the interval $|x - x_0| < R$ for some $R > 0$.

Examples.

1. Find a series solution of the equation

$$y'' + y = 0, \quad -\infty < x < \infty.$$

Solution. $y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n!)} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = a_0 \cos x + a_1 \sin x$

2. **Hermite's equation.** Consider the second-order differential equation

$$y'' - 2xy' + 2py = 0,$$

where p is a numerical parameter. See Appendix B in your textbook for more discussion of this example.

Solution. We obtain $a_2 = -pa_0$. The general recurrence relation is $a_{n+2} = \frac{-2(p-n)a_n}{(n+1)(n+2)}$. Since a_{n+2} is given in terms of a_n, the a_n's will be determined in steps of 2:

- a_0 determines a_2, which determines a_4, which determines a_6, etc.
- a_1 determines a_3, which determines a_5, which determines a_7, etc.

We obtain

$$\begin{aligned} a_2 &= -pa_0 \\ a_3 &= \frac{-2(p-1)a_1}{2 \cdot 3} \\ a_4 &= \frac{-2(p-2)a_2}{3 \cdot 4} = \frac{2p(p-2)a_0}{3 \cdot 4} \\ a_5 &= \frac{-2(p-3)a_3}{4 \cdot 5} = \frac{2(p-1)(p-3)a_1}{3 \cdot 4 \cdot 5}, \end{aligned}$$

etc.

(a) Find the particular solution of Hermite's equation with $p = 0$ that satisfies $y(0) = 1,\ y'(0) = 0$.

Solution. The particular solution is $y(x) = 1$.

(b) Find the particular solution of Hermite's equation with $p = 1$ that satisfies $y(0) = 0,\ y'(0) = 1$.

Solution. The particular solution is $y(x) = x$.

(c) Find the particular solution of Hermite's equation with $p = 2$ that satisfies $y(0) = 1,\ y'(0) = 0$.

Solution. The particular solution is $y(x) = 1 - 2x^2$.

In general, it can be shown that if p is a positive even integer and $(y(0), y'(0)) = (1, 0)$, then the resulting series solution has only finitely many nonzero terms. The same holds if p is a positive odd integer and $(y(0), y'(0)) = (0, 1)$. These solutions are called the **Hermite polynomials** $H_p(x)$. We have shown that the first three Hermite polynomials are $H_0(x) = 1,\ H_1(x) = x,\ H_2(x) = 1 - 2x^2$, and we can easily construct many more.

3. Find a series solution in powers of x of **Airy's equation**

$$y'' - xy = 0, \quad -\infty < x < \infty.$$

Solution. The general recurrence relation is $(n+2)(n+1)a_{n+2} = a_{n-1}$. Thus, we obtain

$$a_{3n+2} = 0$$

,

$$a_{3n} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-1)(3n)},$$

and

$$a_{3n+1} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3n)(3n+1)}.$$

The general solution is

$$y(x) = a_0 \left[1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \cdots + \frac{x^{3n}}{2 \cdot 3 \cdot 4 \cdots (3n-1)(3n)} + \cdots \right]$$
$$+ a_1 \left[x + \frac{x^4}{3 \cdot 4} + \frac{x^7}{3 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{x^{3n+1}}{3 \cdot 4 \cdots (3n)(3n+1)} + \cdots \right].$$

4. Find a solution of Airy's equation in powers of $x - 1$.

Solution. The general recurrence relation is $(n+2)(n+1)a_{n+2} = a_n + a_{n-1}$. The general solution is

$$y(x) = a_0 \left[1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \frac{(x-1)^5}{30} + \cdots \right]$$
$$+ a_1 \left[(x-1) + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \frac{(x-1)^5}{120} + \cdots \right].$$

In general, when the recurrence relation has more than two terms, the determination of a formula for a_n in terms of a_0 and a_1 will be fairly complicated, if not impossible. In this example, such a formula is not readily apparent.