Math 333
Review of Power Series

Finding the general solution of a linear differential equation depends on determining a fundamental set of solutions of the homogeneous equation. So far, we have developed a systematic procedure for constructing fundamental solutions *only if the differential equation has constant coefficients*. To deal with the much larger class of equations that have variable coefficients, we must extend our search for solutions beyond the familiar elementary functions of calculus. The principal tool that we need is the representation of a given function by a power series. The basic idea is similar to the idea that we used in the method of undetermined coefficients: we assume that the solutions of a given differential equation have power series expansions, and then we try to determine the coefficients so as to satisfy the differential equation.

We'll begin by briefly summarizing the basic mathematical results about power series that we need. If you would like more reference material on series, please see me or consult your Calculus B notes and/or textbook.

1. A **power series** centered at $x = x_0$ is a series of the form
$$\sum_{n=0}^{\infty} a_n(x - x_0)^n.$$

2. The power series $\displaystyle\sum_{n=0}^{\infty} a_n(x - x_0)^n$ **converges** at a point x if
$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n(x - x_0)^n$$
exists for that value of x. The power series obviously converges for $x = x_0$. In general, the series may converge for all x, or may converge for some values of x and not for others.

3. The series $\displaystyle\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is said to **converge absolutely** at a point x if the series
$$\sum_{n=0}^{\infty} |a_n(x - x_0)^n| = \sum_{n=0}^{\infty} |a_n||x - x_0|^n$$
converges. If a series converges absolutely, then the series converges. However, the converse is not necessarily true.

4. The **ratio test** is a useful test for the absolute convergence of a series. Consider an arbitrary series $\sum_{n=0}^{\infty} c_n$. Let L denote the limit

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|.$$

The ratio test says the following:

- If $L < 1$, then the series converges absolutely (and hence, converges).
- If $L > 1$, then the series diverges.
- If $L = 1$, then the test is inconclusive.

Example 1. For which values of x does the power series

$$\sum_{n=1}^{\infty}(-1)^{n+1}n(x-2)^n$$

converge?

Solution. The series converges for $1 < x < 3$.

5. For a given power series

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n,$$

there are only three possibilities:

(a) The series converges only when $x = x_0$.
(b) The series converges for all x.
(c) There is a positive number R such that the series converges if $|x - x_0| < R$ and diverges if $|x - x_0| > R$.

The number R in case 3 is called the **radius of convergence** of the power series. The radius of convergence is $R = 0$ in case 1 and $R = \infty$ in case 2. The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges. In case 1, the interval consists of just the single point x_0. In case 2, the interval is $(-\infty, \infty)$. In case 3, there are four possibilities for the interval of convergence:

$$(x_0 - R, x_0 + R), \quad (x_0 - R, x + 0 + R], \quad [x_0 - R, x_0 + R), \quad [x_0 - R, x_0 + R].$$

Example 2. Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{x^n}{n2^n}.$$

Solution. The interval of convergence is $[-2, 2)$ and the radius of convergence is 2.

6. Suppose that $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ and $\sum_{n=0}^{\infty} b_n(x-x_0)^n$ converge to $f(x)$ and $g(x)$, respectively, for $|x-x_0|<R$, $R>0$. The series can be added or subtracted termwise, and the resulting series converges at least for $|x-x_0|<R$:

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x-x_0)^n.$$

The series can be formally multiplied, and the resulting series converges at least for $|x-x_0|<R$:

$$f(x)g(x) = \sum_{n=0}^{\infty} c_n(x-x_0)^n,$$

where $c_n = a_0b_n + a_1b_{n-1} + \cdots + a_nb_0$.

7. If the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ has radius of convergence $R>0$, then the function

$$S(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$$

is both differentiable and integrable, and

- $S'(x) = \sum_{n=1}^{\infty} na_n(x-x_0)^{n-1} = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \cdots$, and the radius of convergence of the series $S'(x)$ is R.
- $\int S(x)\,dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-x_0)^{n+1}}{n+1} = C + a_0(x-x_0) + a_1\frac{(x-x_0)^2}{2} + a_2\frac{(x-x_0)^3}{3} + \cdots$, and the radius of convergence of the series $\int S(x)\,dx$ is R.

8. If a function $f(x)$ has a power series expansion

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n,$$

then the value of a_n is given by

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

The series is called the **Taylor series** for the function f about $x=x_0$. A function f that has a Taylor series expansion about $x=x_0$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n,$$

with radius of convergence $R > 0$ is said to be **analytic** at $x = x_0$. A Taylor series centered at $x_0 = 0$ is called a **Maclaurin series**.

9. If $\sum_{n=0}^{\infty} a_n(x - x_0)^n = \sum_{n=0}^{\infty} b_n(x - x_0)^n$, then $a_n = b_n$ for $n = 0, 1, 2, 3, \ldots$.

Example 3. Suppose that

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n$$

for all x. Determine what this implies about the coefficients a_n.

Solution. $a_n = \dfrac{a_0}{n!}$, $n = 1, 2, 3, \ldots$. Thus, $\sum_{n=0}^{\infty} a_n x^n = a_0 e^x$.

Example 4. Determine the a_n so that the equation

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

is satisfied. Identify the function represented by the series $\sum_{n=0}^{\infty} a_n x^n$.