Introduction to Series

Definition. A **series** is an infinite sum of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_2 + \ldots + a_n + a_{n+1} + \cdots$$

or

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_2 + \ldots + a_n + a_{n+1} + \cdots$$

.

Example 1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$

Example 2. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+1} = 1 + \frac{1}{2} + \frac{1}{5} + \cdots$

We will be interested in thinking about the following questions:

- What does it mean to add up infinitely many numbers?
- Which series converge (i.e. add up to a finite number)? Which series diverge (i.e. go to infinity)?
- How do series relate to functions and the other topics studied in Calculus?

Example 3. Use a geometric argument to show that

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 2.$$

Some definitions and terminology. Let

$$\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + a_3 + \cdots$$

be an infinite series. The summand a_k is called the k-th **term** of the series. The sum of the first n terms of the series is called the n-th **partial sum** of the series, and is denoted by S_n:

$$S_n = a_0 + a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=0}^{n} a_k.$$

The definition of **convergence** of an infinite series involves the partial sums S_n.

Definition. If

$$\lim_{n\to\infty} S_n = S$$

for some finite number S, then the series

$$\sum_{k=0}^{\infty} a_k$$

converges to the limit S. Otherwise, the series **diverges**.

Example 4. Use the definition above to show that

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 2.$$

Example 5. Telescoping series. Use the definition above to show that

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$$

converges to 1.

Example 6. Discuss the series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

This series is called the **harmonic series**. Can we use the definition to determine whether or not the series converges?

The n-th term test for divergence. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=0}^{\infty} a_n$ diverges.

Example 7. Does the series

$$\sum_{k=1}^{\infty} \frac{2k^2 - 3k + 1}{k^2 + 4}$$

converge or diverge?

Example 8. Does the series

$$\sum_{k=1}^{\infty} (-1)^k$$

converge or diverge?