Convergence and Divergence of Series

- **The n-th term test for divergence.** If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=0}^{\infty} a_n$ diverges.

  Example. Since
  $$\lim_{k\to\infty} \frac{2k^2-3k+1}{k^2+4} = 2 \neq 0,$$
  the series
  $$\sum_{k=1}^{\infty} \frac{2k^2-3k+1}{k^2+4}$$
  diverges.

- **Telescoping series.** We can use partial sums to determine whether or not a given telescoping series converges. Keep in mind that usually (but not always), you will use partial fractions to identify a series as a telescoping series.

  Example. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$. The n-th partial sum is
  $$S_n = 1 - \frac{1}{n+1}.$$
  Thus, since $\lim_{n\to\infty} S_n = 1$, the series converges to 1.

- **Geometric series.** The geometric series
  $$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots$$
  :
  - converges to $\frac{a}{1-r}$ if $|r| < 1$
  - diverges if $|r| \geq 1$

  Example. $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$ is a geometric series with $a = 1$ and $r = 1/2$. Since $|1/2| < 1$, the series converges to $\frac{1}{1-\frac{1}{2}} = 2$.

- **Integral test for positive series.** Suppose that, for all $x \geq 1$, the function $a(x)$ is continuous, positive, and decreasing. Let $a_k = a(k)$ for all integers $k \geq 1$. Consider the series and the integral

$$\sum_{k=1}^{\infty} a_k \text{ and } \int_1^{\infty} a(x)\,dx.$$

  - If either diverges, so does the other.
  - If either converges, so does the other. Note: the integral test does not tell us what value the series converges to; the test just tells us that the series converges if the corresponding integral converges.

Example. Since the integral $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$, diverges, the harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ also diverges.

- **The p-test for series.** The p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$

  - converges if $p > 1$. Note: the p-test does not tell us what value the series converges to; the test just tells us that the series converges if $p > 1$.
  - diverges if $p \leq 1$

- **Comparison test for non-negative series.** Consider two series Σa_k and Σb_k. Suppose that

$$0 \leq a_k \leq b_k$$

for all k.

  - If Σb_k converges, so does Σa_k.
  - If Σa_k diverges, so does Σb_k.

Example. Since

$$\frac{1}{k^2+2} \leq \frac{1}{k^2}$$

for all k, and since $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges ($p = 2 > 1$), the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ also converges.

- **Ratio test.** Let
$$L = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|.$$
  - If $L < 1$, then Σa_k converges. Note: the ratio test does not tell us what value the series converges to; the test just tells us that the series converges if $L < 1$.
  - If $L > 1$, then Σa_k diverges.
  - If $L = 1$, then the test gives no information, and other techniques must be used.

The Ratio test is often useful for series in which the index k appears in an exponent or a factorial.

Example. Consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k!}$. Since
$$\lim_{k\to\infty} \left| \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} \right| = 0 < 1,$$
the series converges.