# Math 333
# Tuesday, April 15, 2008
# Graphing Series Solutions

We have been studying series solutions of equations of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{1}$$

where $P(x)$, $Q(x)$, $R(x)$ are polynomial functions of $x$ (or, in general, analytic functions of $x$). Our strategy is to look for solutions of Eqn. (1) of the form

$$y = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + \cdots = \sum_{n=0}^{\infty} a_n(x-x_0)^n,$$

around an ordinary point $x_0$, and assume that the series converges in the interval $|x-x_0| < R$ for some $R > 0$. Next, we'll consider how to graphically represent series solutions, and discuss the role of the center $x_0$ of the series solution.

**Example.** Previously, we have used series techniques to find the general solution of the differential equation

$$y'' + y = 0.$$

We have shown that the general solution is given by the series

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n!)} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

centered at $x_0 = 0$. Using known Maclaurin series, we identified the solution as

$$y(x) = a_0 \cos x + a_1 \sin x.$$

Let $y(0) = a_0 = 1$ and $y'(0) = a_1 = 1$ and graph partial sums (polynomial approximations)

$$\sum_{n=N}^{\infty} \frac{(-1)^n}{(2n!)} x^{2n} + \sum_{n=N}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

of the solution for various values of $N$ on the same set of axes as the known exact solution

$$\cos x + \sin x.$$

See the Maple file *SeriesPlots* posted on the Course Schedule page and on the P: drive for help with the syntax. In Figure , we illustrate the polynomial approximations for $N = 2$ and $N = 5$ on the same set of axes as the exact solution $y = \cos x + \sin x$. Observe that as the number of terms in the partial sums increases, the interval over which the approximation is satisfactory becomes larger, and for each $x$ in this

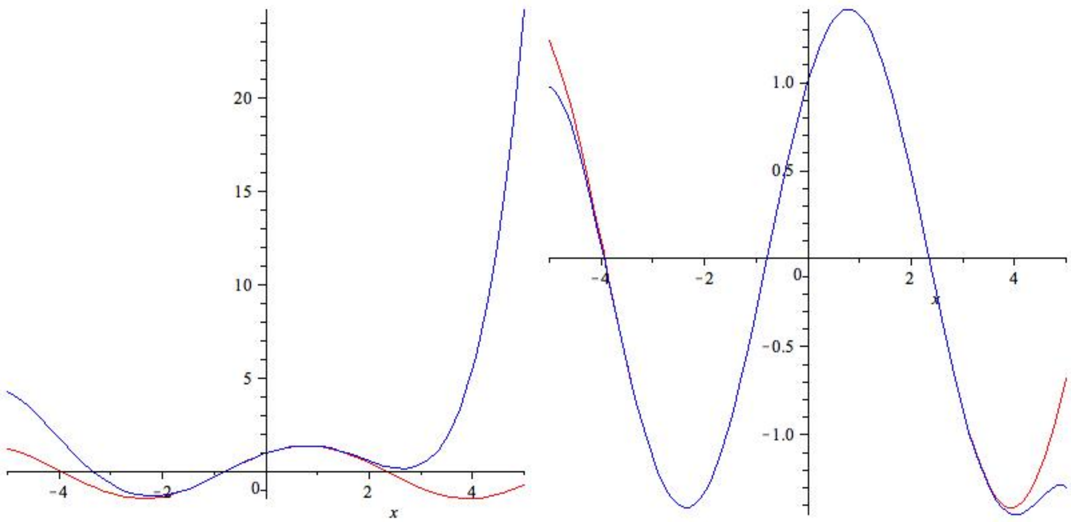


Figure 1: Partial sum approximations with $N = 2$ (left) and $N = 5$ (right).

interval, the accuracy of the approximation improves. However, the truncated power series provides only a local approximation of the solution in a neighborhood of the point $x_0 = 0$; a truncated power series cannot adequately represent the solution for large $|x|$.

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In general, when we use series techniques to obtain a series solution

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

centered at $x = x_0$, we must use partial sums of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_N(x - x_0)^N$$

to obtain graphical and/or numerical representations of the solution. It is important to remember that such partial sums provide only a local approximation of the solution in a neighborhood of the point $x_0$; a truncated power series cannot adequately represent the solution when $|x - x_0$ is large. As $N$ (the number of terms used in the partial sum) increases, the interval over which the approximation is satisfactory becomes larger.