Sequences (continued)
The Squeeze Theorem
The Monotonic Sequence Theorem

The Squeeze Theorem. Suppose that $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences such that

$$a_n \leq b_n \leq c_n$$

for all $n \geq 1$ and that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L.$$

Then

$$\lim_{n\to\infty} b_n = L.$$

Example 1. Consider the sequence $\{a_n\}$ defined by

$$a_n = \frac{\sin n}{n}.$$

Find $\lim_{n\to\infty} a_n$. Does $\{a_n\}$ converge or diverge?

Example 2. Consider the sequence $\{a_n\}$ defined by

$$a_n = \frac{(-1)^n}{n^2+2}.$$

Find $\lim_{n\to\infty} a_n$. Does $\{a_n\}$ converge or diverge?

The Monotonic Sequence Theorem.

1. Suppose that the sequence $\{a_n\}$ is **monotone increasing**, i.e. $a_{n+1} \geq a_n$ for all n, and that a_n is **bounded above** by some number A, i.e. $a_n \leq A$ for all n. Then $\{a_n\}$ converges to some finite limit a with $a \leq A$.

2. Suppose that the sequence $\{a_n\}$ is **monotone decreasing**, i.e. $a_{n+1} \leq a_n$ for all n, and that a_n is **bounded below** by some number B, i.e. $a_n \geq B$ for all n. Then $\{a_n\}$ converges to some finite limit b with $b \geq B$.

Example 1. Consider the sequence $\{a_n\}$ defined by

$$a_1 = 1, \quad a_{n+1} = 3 - \frac{1}{a_n}.$$

This is an example of a **recursively-defined sequence**. Show that the sequence $\{a_n\}$ is monotone increasing and bounded above by 3. Deduce that $\{a_n\}$ is convergent.