Sequences

Definition. A **sequence** is an infinite list of numbers

$$a_1,\ a_2,\ a_3, \ldots, a_k,\ a_{k+1}, \ldots.$$

Individual entries are called the **terms** of the sequence.
Notation. We will use the following standard notation for sequences:

$$\{a_n\}_{n=1}^{\infty} = \{a_n\} = a_1,\ a_2,\ a_3,\ , \ldots$$

Example 1. Consider the sequence $\{a_n\}$ defined by

$$a_n = \frac{1}{n}.$$

What is a_1? What is a_2? What is a_5? What is a_{100}?

Definition. A sequence a_n has **limit** L if

$$\lim_{n\to\infty} a_n = L$$

or

$$a_n \to L \text{ as } n \to \infty.$$

This means that we can make the terms a_n *arbitrarily close* to L by making n sufficiently large. If $\lim_{n\to\infty} a_n$ exists (and is equal to a real number), we say that the sequence $\{a_n\}$ **converges** (or is **convergent**). Otherwise, we say that the sequence **diverges** (or is **divergent**).

Example 2. Consider the sequence $\{a_n\}$ defined by

$$a_n = \frac{1}{n}.$$

Find $\lim_{n\to\infty} a_n$. Does the sequence $\{a_n\}$ converge or diverge?

Example 3. Consider the sequence $\{a_n\}$ defined by

$$a_n = n + 2.$$

Find $\lim_{n\to\infty} a_n$. Does the sequence $\{a_n\}$ converge or diverge?

Example 4. Consider the sequence $\{a_n\}$ defined by

$$a_n = (-1)^n.$$

Find $\lim_{n\to\infty} a_n$. Does the sequence $\{a_n\}$ converge or diverge?

Example 5. Consider the sequence $\{a_n\}$ defined by

$$a_n = \left(\frac{1}{4}\right)^n.$$

Find $\lim_{n\to\infty} a_n$. Does the sequence $\{a_n\}$ converge or diverge?

Example 6. Consider the sequence $\{a_n\}$ defined by

$$a_n = 4^n.$$

Find $\lim_{n\to\infty} a_n$. Does the sequence $\{a_n\}$ converge or diverge?

Example 7. For what values of r is the sequence $\{a_n\}$ defined by

$$a_n = r^n$$

convergent?

The following theorems are useful for evaluating limits.

Theorem. Suppose that $f(x)$ is a function such that $f(n) = a_n$ for all integers $n \geq 1$. If

$$\lim_{x\to\infty} f(x) = L,$$

then

$$\lim_{n\to\infty} a_n = L.$$

Theorem: Algebra with limits. Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent sequences with

$$\lim_{n\to\infty} a_n = L \text{ and } \lim_{n\to\infty} b_n = M.$$

Let c be any real constant. Then:

- $\lim_{n\to\infty} (a_n + b_n) = L + M$
- $\lim_{n\to\infty} (a_n - b_n) = L - M$
- $\lim_{n\to\infty} (ca_n) = cL$
- $\lim_{n\to\infty} (a_n b_n) = LM$
- $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{L}{M}$ if $M \neq 0$

Example 8. Consider the sequence $\{a_n\}$ defined by

$$a_n = \frac{2^n}{n^2}.$$

Find $\lim_{n\to\infty} a_n$. Does the sequence $\{a_n\}$ converge or diverge?

Example 9. Consider the sequence $\{a_n\}$ defined by

$$a_n = \frac{1}{n} + \frac{3n}{n+1}.$$

Find $\lim_{n\to\infty} a_n$. Does the sequence $\{a_n\}$ converge or diverge?

Example 10. Consider the sequence $\{a_n\}$ defined by

$$a_n = \frac{\ln n}{n+2}.$$

Find $\lim_{n\to\infty} a_n$. Does the sequence $\{a_n\}$ converge or diverge?

Example 11. Consider the sequence $\{a_n\}$ defined by

$$a_n = \left(1 + \frac{2}{n}\right)^n.$$

Find $\lim_{n\to\infty} a_n$. Does the sequence $\{a_n\}$ converge or diverge?