Math 333
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Introduction to Second Order Linear Equations

Definition. A **second order ordinary differential equation** is an equation of the form

$$\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt}). \tag{1}$$

An **initial-value problem** containing a second order differential equation consists of a differential equation of the form in Eqn. (1) together with a *pair* of initial conditions

$$y(t_0) = y_0 \text{ and } y'(t_0) = y_0'.$$

Observe that the initial conditions for a second order differential equation prescribe not only a particular point (t_0, y_0) through which the graph of the solution must pass, but also the slop y_0' of the graph at that point. It is reasonable to expect that two initial conditions are needed for a second order differential equation because, roughly speaking, two integrations are required to find a solution, and each integration introduces an arbitrary constant.

Definition. A second order differential equation is **linear** if Eqn. (1) has the form

$$f(t, y, \frac{dy}{dt}) = g(t) - p(t)\frac{dy}{dt} - q(t)y, \tag{2}$$

i.e. if f is linear in y and y'. Thus, a linear second order differential equation is one of the form

$$y''(t) + p(t)y' + q(t)y = g(t). \tag{3}$$

Otherwise, the differential equation is nonlinear.

Definition. A second order linear differential equation is **homogeneous** if the $g(t)$ term in Eqn. (3) is zero for all t. Thus, a homogeneous second order linear differential equation is one of the form

$$y''(t) + p(t)y' + q(t)y = 0. \tag{4}$$

Otherwise, the differential equation is nonhomogeneous. Later in this course, we'll show that once the homogeneous equation has been solved, it is always possible to solve the corresponding nonhomogeneous equation, or at least to express the solution in terms of an integral. Thus, the problem of solving the homogeneous linear second order differential equation is the more fundamental one.

Homogeneous Equations with Constant Coefficients

To begin our discussion, we'll consider homogeneous linear second order differential equations with **constant coefficients**. These are DE's of the form

$$ay'' + by' + cy = 0, \tag{5}$$

where a, b, and c are constants. We assume that $a \neq 0$ since otherwise, we have a first order linear differential equation (which we already know how to solve). It turns out that Eqn. (5) can *always* be solved in terms of the elementary functions of calculus. However, it is usually much more difficult to solve the general homogeneous linear second order differential equation if the coefficients are not constants, and we will consider that case later in the course. Let's start with some discussion of the general homogeneous linear second order DE with constant coefficients:

$$ay'' + by' + cy = 0,$$

where a, b, and c are arbitrary real constants with $a \neq 0$. Based on our experience with first order linear DE's, it is natural for us to guess that a function of the form $y(t) = e^{rt}$ might be a solution, where r is a parameter to be determined. If we guess a solution of the form $y = e^{rt}$, then $y' = re^{rt}$ and $y'' = r^2e^{rt}$. Thus, the DE becomes

$$(ar^2 + br + c)e^{rt} = 0. \tag{6}$$

Since $e^{rt} \neq 0$, Eqn. (6) is rewritten as

$$ar^2 + br + c = 0. \tag{7}$$

Eqn. (7) is called the **characteristic equation** of the DE, and the roots of the characteristic equation are called the **characteristic values** or **eigenvalues** of the DE.

For now, let's assume that the roots of the characteristic equation (7) are *real* and *distinct*. We'll consider the other cases (a repeated real root or complex conjugate roots) later. So, suppose that r_1 and r_2 are roots of the characteristic equation (7). Then

$$y_1(t) = e^{r_1 t}$$

and

$$y_2(t) = e^{r_2 t}$$

are solutions of the DE

$$ay'' + by' + cy = 0.$$

Moreover, for any values of the arbitrary constants k_1 and k_2, the function

$$y(t) = k_1e^{r_1 t} + k_2e^{r_2 t}$$

is also a solution. Thus, we have constructed a function with two arbitrary constants that satisfies the original differential equation $ay'' + by' + cy = 0$. Since the function satisfies the DE and contains two arbitrary constants, we expect that it is the general solution of the homogeneous linear second order DE with constant coefficients. Indeed, it can be shown that this function $y(t)$ is the general solution. This is part of a more general theorem that we will study later. For now, let's summarize what we've learned so far.

Theorem. Consider the homogeneous linear second order DE with constant coefficients

$$ay'' + by' + cy = 0.$$

Suppose that the characteristic equation

$$ar^2 + br + c = 0$$

has two real distinct roots, r_1 and r_2. Then

$$y(t) = k_1 e^{r_1 t} + k_2 e^{r_2 t},$$

where k_1 and k_2 are arbitrary constants, is the general solution of the DE $ay'' + by' + cy = 0$.

Examples

Example 1. Find the general solution of $y'' + 5y' + 6y = 0$. Describe the behavior of the solutions as $t \to \infty$. **Solution.** The general solution is $y(t) = c_1 e^{-2t} + c_2 e^{-3t}$. As $t \to \infty$, $y(t) \to 0$.

Example 2. Solve the initial-value problem $y'' + 5y' + 6y = 0$, $y(0) = 2$, $y'(0) = 3$. Describe the behavior of the solution as $t \to \infty$. **Solution.** $y(t) = 9e^{-2t} - 7e^{-3t}$. As $t \to \infty$, $y(t) \to 0$.

Example 3. Solve the initial-value problem $4y'' - 8y' + 3y = 0$, $y(0) = 2$, $y'(0) = 1/2$. Describe the behavior of the solution as $t \to \infty$. **Solution.** $y = -\frac{1}{2}e^{3t/2} + \frac{5}{2}e^{t/2}$. As $t \to \infty$, $y(t) \to -\infty$.