Math 112
Integrals as Riemann Sums

Suppose that we want to find

$$\int_a^b f(x)\,dx$$

using Riemann sums. Remember that $\displaystyle\int_a^b f(x)\,dx$ is the signed area of the region bounded by the graph of $f(x)$ and the vertical lines $x = a$ and $x = b$.

1. Start by dividing the interval $[a, b]$ into n equal subintervals. Note that the width of each subinterval is
$$\Delta x = \frac{b-a}{n}.$$
2. Next, choose an x-value in each subinterval. We'll let x_i denote the x-value that we've chosen in the i-th subinterval. For example, x_i might be the left endpoint of the i-th subinterval, or it might be the right endpoint of the i-the subinterval, or it might be the midpoint of the i-th subinterval, or it might be another point altogether.
3. In each subinterval, draw the rectangle with height $f(x_i)$.
4. Add up the areas of all of the rectangles. This gives an *approximation* for the integral $\displaystyle\int_a^b f(x)\,dx$. In symbols:
$$\int_a^b f(x)\,dx \approx [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$
Using sigma notation, we can write this as:
$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i)\Delta x.$$
Remember that $\Delta x = \frac{b-a}{n}$.
5. To find the *exact* value for the integral (i.e. the area), let the number of rectangles used go to infinity:
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$
where x_i is any point in the i-th sub-interval. For consistency and ease of notation, we will typically take x_i to be the right (or left, if you prefer) endpoint of the i-th subinterval. Using right endpoints, we can rewrite the previous Riemann sum as:
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + \frac{i(b-a)}{n}\right)\frac{b-a}{n},$$

where we have used that $\Delta x = \frac{b-a}{n}$. If you want to use left endpoints, then the Riemann sum becomes:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + \frac{(i-1)(b-a)}{n}\right)\frac{b-a}{n}.$$

Note that these two expressions will give you the same result for the integral! When you are working with sums of this form, remember that i is your index of summation, so anything that does not involve i (e.g. constants or expressions involving n) can be pulled out of the sum.

Example. Evaluate $\displaystyle\int_1^4 x\,dx$ by computing the limit of right sums R_n.