Math 333
Repeated Roots of the Characteristic Equation

Let's return to the second order linear homogeneous differential equation with constant coefficients:

$$ay'' + by' + cy = 0. \tag{1}$$

First, let's summarize our previous results. Recall that if the roots r_1 and r_2 of the characteristic equation

$$ar^2 + br + c = 0$$

are real and different, then the general solution of Eqn. (1) is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

If the roots $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$ are complex conjugates, then the general solution of Eqn. (1) is

$$y(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t).$$

Next, we'll consider the case in which the characteristic equation

$$ar^2 + br + c = 0$$

has a repeated real root

$$r_1 = r_2 = \frac{-b}{2a}$$

Then both roots yield the same solution

$$y_1(t) = e^{-bt/2a},$$

and it is not clear how to obtain a second solution. Let's start with an example.

Example 1. Solve the differential equation

$$y'' + 4y' + 4y = 0.$$

Solution. The characteristic equation is

$$r^2 + 4r + 4 = (r + 2)^2 = 0,$$

so $r = -2$ is a double root. Thus one solution of the differential equation is

$$y_1(t) = e^{-2t}.$$

To find the general solution of the differential equation, we need to find a second solution that is not a multiple of y_1. Recall from the Linearity Principle that since

$y_1(t)$ is a solution of the differential equation, so is $cy_1(t)$ for any constant c. The basic idea that we'll use here is to try to generalize this observation by replacing c with a function $v(t)$, and then to try to determine $v(t)$ so that $v(t)y_1(t)$ is also a solution of the differential equation. To carry out this method, we'll substitute $v(t)y_1(t)$ in the differential equation and use the resulting equation to find $v(t)$. Starting with

$$y = v(t)y_1(t) = v(t)e^{-2t},$$

we obtain

$$y' = v'(t)e^{-2t} - 2v(t)e^{-2t}$$

and

$$y'' = v''(t)e^{-2t} - 4v'(t)e^{-2t} + 4v(t)e^{-2t}.$$

Substituting these expressions into the differential equation and collecting terms, we obtain:

$$[v''(t) - 4v'(t) + 4v(t) + 4v'(t) - 8v(t) + 4v(t)]e^{-2t} = 0.$$

This simplifies to

$$v''(t) = 0.$$

Thus

$$v'(t) = k_1$$

and

$$v(t) = k_1 t + k_2.$$

Thus we obtain a second solution

$$v(t)y_1(t) = (k_1 t + k_2)e^{-2t}.$$

Thus the solution $y(t)$ is of the form

$$y(t) = k_1 t e^{-2t} + k_2 e^{-2t}.$$

We have thus written the general solution as a linear combination of two different solutions y_1 and y_2 :

$$y_1(t) = e^{-2t} \text{ and } y_2(t) = te^{-2t}.$$

To verify that y_1 and y_2 form a fundamental set of solutions, we calculate their Wronskian:

$$W(y_1, y_2) = e^{-4t} \neq 0.$$

Thus the general solution of the original differential equation is

$$y(t) = k_1 t e^{-t/2} + k_2 e^{-2t}.$$

Summary. The procedure used in the previous example can be generalized as follows. Consider the second-order linear homogeneous differential equation in Eqn. (1) with constant coefficients:

$$ay'' + by' + cy = 0.$$

Suppose that the characteristic equation

$$ar^2 + br + c = 0$$

has a repeated double root, i.e. we assume that the coefficients satisfy

$$b^2 - 4ac = 0.$$

Thus,

$$y_1(t) = e^{-bt/2a}$$

is a solution. Then we assume that

$$y(t) = v(t)y_1(t) = v(t)e^{-bt/2a} \tag{2}$$

is a solution, and substitute in Eqn. (1) to determine the function $v(t)$. We compute the derivatives:

$$y' = v'(t)e^{-bt/2a} - \frac{b}{2a}v(t)e^{-bt/2a}$$

and

$$y'' = v''(t)e^{-bt/2a} - \frac{b}{a}v'(t)e^{-bt/2a} + \frac{b^2}{4a^2}v(t)e^{-bt/2a}.$$

Then, substituting in Eqn. (1), we obtain

$$v''(t) = 0.$$

Thus,

$$v'(t) = c_1$$

and

$$v(t) = k_1 t + k_2.$$

Thus,

$$y(t) = k_1 te^{-bt/2a} + k_2 e^{-bt/2a},$$

and we have written y as a linear combination of the two solutions

$$y_1(t) = e^{-bt/2a} \text{ and } y_2(t) = te^{-bt/2a}.$$

Since

$$W(y_1, y_2) = e^{-bt/a} \neq 0,$$

we conclude that the general solution of the differential equation given by Eqn. (1) is

$$y(t)k_1 te^{-bt/2a} + k_2 e^{-bt/2a}.$$

Example 2. Find the solution of the initial-value problem

$$y'' - y' + 0.25y = 0, \quad y(0) = 2, \quad y'(0) = 1/3.$$

Solution. $y(t) = 2e^{t/2} - \frac{2}{3}te^{t/2}$