Regular Singular Points

We will now consider the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{1}$$

in the neighborhood of a singular point x_0. Recall that if the functions P, Q, and R are polynomials with no common factors, then the singular points of Eqn. (1) are the points for which $P(x) = 0$.

Example 1. The point $x = 0$ is a singular point of the differential equation

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0.$$

All other points are ordinary points.

Unfortunately, we cannot, in general, use the series methods that we developed previously to solve Eqn. (1) in the neighborhood of a singular point x_0 because the solution of Eqn. (1) often cannot be represented by a Taylor series in powers of $x - x_0$.

Since the singular points of a differential equation are usually few in number, we might ask whether we can simply ignore them, especially since we already know how to construct solutions about ordinary points. However, the singular points often determine the principal features of the solution to a much larger extent than we might at first suspect. In the neighborhood of a singular point, the solution often becomes large in magnitude or experiences rapid change in magnitude. Thus, the behavior of a physical system modeled by a differential equation frequently is most interesting in the neighborhood of a singular point. Often, geometric singularities in the physical situation being modeled, such as corners or sharp edges, lead to singular points in the corresponding differential equation. Thus, although at first we might want to avoid the few points where a differential equation is singular, it is precisely at these points that it is necessary to study the solution most carefully.

In general, it is impossible to characterize the behavior of the solutions of Eqn. (1) in the neighborhood of a singular point x_0. It may be that there are two linearly independent solutions of Eqn. (1) that remain unbounded as $x \to x_0$; there may be only one, with the other becoming unbounded as $x \to x_0$; or both solutions may become unbounded as $x \to x_0$. The examples below illustrate these possibilities. Finally, if Eqn. (1) does have solutions that become unbounded as $x \to x_0$, it is often important to determine how these solutions behave as $x \to x_0$. For example, does $y(x) \to \infty$ in the same way as $\dfrac{1}{x - x_0}$, or as $\dfrac{1}{(x - x_0)^2}$, or in some other way?

Example 2. Consider the differential equation

$$x^2y'' - 2y = 0.$$

The differential equation has a singular point at $x = 0$. We can verify by direct substitution that

$$y_1(x) = x^2 \text{ and } y_2(x) = \frac{1}{x}$$

are two linearly independent solutions of the differential equation. Thus, in any interval not containing 0, the general solution is

$$y(x) = c_1 x^2 + c_2 \frac{1}{x}.$$

If $c_2 = 0$, then the solution is bounded as $x \to 0$. Notice that the solution $y_2(x) = \frac{1}{x}$ does not have a Taylor series expansion about $x_0 = 0$; thus, the series techniques developed previously would fail to find the solution $y_2(x) = \frac{1}{x}$.

Example 3. Consider the differential equation

$$x^2 y'' - 2xy' + 2y = 0.$$

The differential equation has a singular point at $x_0 = 0$. We can verify by direct substitution that

$$y_1(x) = x \text{ and } y_2(x) = x^2$$

are two linearly independent solutions of the differential equation. Thus, the general solution is

$$y(x) = c_1 x + c_2 x^2.$$

Notice that both y_1 and y_2 have a Taylor series expansion centered at $x_0 = 0$. However, it is still not proper to pose an initial-value problem with initial conditions at $x = 0$. It is impossible to satisfy arbitrary initial conditions at $x = 0$ since any linear combination of x and x^2 is zero at $x = 0$.

Example 4. Consider the differential equation

$$x^2 y'' + 5xy' + 3y = 0.$$

The differential equation has a singular point at $x_0 = 0$. We can verify by direct substitution that

$$y_1(x) = \frac{1}{x} \text{ and } y_2(x) = \frac{1}{x^2}$$

are two linearly independent solutions of the differential equation. Neither solution has a Taylor series expansion around $x_0 = 0$. Indeed, every (nonzero) solution of the differential equation becomes unbounded as $x \to 0$.

In our continued discussion of solutions of Eqn. (1) near a singular point x_0, we will need the following terminology.

Definition. If

$$\lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)} \text{ is finite}$$

and

$$\lim_{x\to x_0}(x-x_0)^2\frac{R(x)}{P(x)} \text{ is finite },$$

then we say that x_0 is a **regular singular point** of Eqn. (1). Any singular point of Eqn. (1) that is not a regular singular point is called an **irregular singular point**. We will develop techniques for solving Eqn. (1) in the neighborhood of a regular singular point.

Example 5. Consider the Legendre equation

$$(1-x^2)y''-2xy'+\alpha(\alpha+1)y=0,$$

where α is a numerical parameter. The singular points $x=\pm1$ are both regular singular points.

Example 6. Consider the differential equation

$$2x(x-2)^2y''+3xy'+(x-2)y=0.$$

The singular points are $x=0$ and $x=2$; $x=0$ is a regular singular point and $x=2$ is an irregular singular point.

Example 7. Consider the differential equation

$$\left(x-\frac{\pi}{2}\right)^2y''+(\cos x)y'+(\sin x)y=0.$$

The singular point $x=\pi/2$ is a regular singular point.