Math 333
Tuesday, February 26, 2008
Reduction of Order

Recall the method that we used to construct a second solution of the differential equation $ay'' + by' + cy = 0$. We knew one solution $y_1(t)$, and we guessed a solution of the form

$$y(t) = v(t)y_1(t),$$

and then used the differential equation to determine the function $v(t)$ so that $y(t)$ is a solution of the differential equation.

This technique is more generally applicable. Consider the general second-order linear homogeneous differential equation

$$y'' + p(t)y' + q(t)y = 0. \tag{1}$$

Suppose that one solution, $y_1(t)$, is known. Assume that $y_1(t)$ is not identically zero. To find a second solution, let

$$y = v(t)y_1(t).$$

Then

$$y' = v'y_1 + vy_1'$$

and

$$y'' = v''y_1 + 2v'y_1' + vy_1''.$$

Substituting for y, y', and y'' in Eqn. (1) and collecting terms, we obtain:

$$y_1v'' + (2y_1' + p(t)y_1)v' + (y_1'' + p(t)y_1' + q(t)y_1)v = 0. \tag{2}$$

Since y_1 is a solution of the differential equation in Eqn. (1), the coefficient of v in Eqn. (2) is zero. Thus, Eqn. (2) becomes:

$$y_1v'' + (2y_1' + p(t)y_1)v' = 0. \tag{3}$$

This is a first-order **separable** linear differential equation for the function v'. Thus, we can find v' by separating variables and integrating both sides. Then, we can find v by integrating. Finally, we obtain a solution $y(t)$ of the original differential equation:

$$y(t) = v(t)y_1(t).$$

This technique is called the **reduction of order** method because we reduce the original second order differential equation for y to a first-order differential equation for v'.

Example. Consider the differential equation

$$2t^2y'' + 3ty' - y = 0, \quad t > 0.$$

Show that

$$y_1(t) = t^{-1}$$

is a solution of the differential equation. Then use the method of reduction or order to find the **general solution** of the differential equation.

Solution. Set $y(t) = v(t)t^{-1}$. Then

$$y' = v't^{-1} - vt^{-2}, \quad y'' = v''t^{-1} - 2v't^{-2} + 2vt^{-3}.$$

Substituting in the differential equation and collecting terms, we obtain

$$2tv'' - v' = 0.$$

We can rewrite this as

$$2t\frac{dv'}{dt} = v',$$

or

$$\frac{1}{v'}\,dv' = \frac{1}{2t}\,dt.$$

Integrating both sides, we obtain

$$v'(t) = c_1t^{1/2}.$$

Thus

$$v(t) = \frac{2}{3}c_1t^{1/2} + c_2.$$

Finally,

$$y(t) = k_1t^{1/2} + k_2t^{-1}.$$

Thus we have written the solution $y(t)$ as a linear combination of two solutions $y_1(t) = t^{-1}$ and $y_2(t) = t^{1/2}$. To verify that $y(t)$ is indeed the general solution, we show that y_1 and y_2 form a fundamental set of solutions. To show that y_1 and y_2 form a fundamental set of solutions, we show that $W(y_1, y_2) \neq 0$.